Algebra and Trigonometry
Review Material

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http://atlas.math.vanderbilt.edu/~pscrooke/calculus/calculus.html

Contents

Numbers and Absolute Values

Numbers

Numbers play a very important role in mathematics, since they are among the most basic and useful mathematical objects. Recall that a collection of objects is called a **set**. A set can be described by listing its elements, as in $A = \{2, 4, 6, 8\}$, or by giving a property that completely determines the elements, such as $A = \{x \mid x$ is an even number between 1 and 9$\}$. This way of writing a set is called **set notation**. The sets of numbers we will be using in calculus are the following:

The set $\mathbb{N}$ of **natural numbers** is the set of counting numbers or positive whole numbers:

$$\mathbb{N} = \{1, 2, 3, ..\}$$

The set $\mathbb{Z}$ of **integers** is the set of all positive and negative whole numbers together with zero:

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

The set $\mathbb{Q}$ of **rational numbers** is the set of all quotients of integers with nonzero denominators:

$$\mathbb{Q} = \left\{ \frac{m}{n} \,\middle|\, m, n \text{ are integers with } n \neq 0 \right\}$$

The set $\mathbb{R}$ of **real numbers** includes all of the above numbers, together with all the irrational numbers like $\sqrt{2}, \sqrt[5]{7}, \pi$, and e. Every real number x can be thought of as the (oriented) distance of a point on a line from the origin. So the set of real numbers is in direct correspondence with a line on a plane.
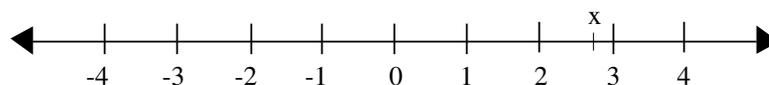

Figure 1.1: The real number line

Because certain subsets of the real line are used quite often, we adopt a special notation for them that is called **interval notation**. The set $\{x \mid 1 \leq x \leq 2\}$ of all real numbers between 1 and 2, including 1 and 2, is denoted by $[1, 2]$. If we want all those

numbers except for 2, we write $[1, 2) = \{x \mid 1 \leq x < 2\}$. In general we have the following notation for certain sets of numbers called **intervals**.

$$[a, b] = \{x \mid a \leq x \leq b\} \qquad \textbf{closed interval}$$
$$[a, b) = \{x \mid a \leq x < b\} \qquad \textbf{half-open interval}$$
$$(a, b] = \{x \mid a < x \leq b\} \qquad \textbf{half-open interval}$$
$$(a, b) = \{x \mid a < x < b\} \qquad \textbf{open interval}$$
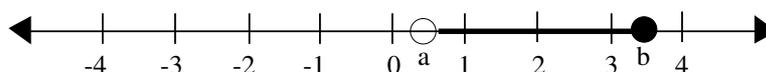

Figure 1.2: The interval $(a, b]$

The hollow circle at $x = a$ on the number line above shows that the interval *does not* contain a, while the filled-in circle at $x = b$ shows that it *does* contain b.

We can consider **infinite intervals** as well:

$$[a, +\infty) = \{x \mid a \leq x\}$$
$$(a, +\infty) = \{x \mid a < x\}$$
$$(-\infty, b] = \{x \mid x \leq b\}$$
$$(-\infty, b) = \{x \mid x < b\}$$
$$(-\infty, +\infty) = \mathbb{R}$$

Note that $-\infty$ and $+\infty$ are never included in the intervals, since they are not numbers.

If A and B are sets, we denote the set of all elements of A together with all elements of B by $A \cup B$, which we read as "A union B."

Example 1:

(a) $\{1, 2, 3, 4\} \cup \{6, 7\} = \{1, 2, 3, 4, 6, 7\}$

(b) $\{1, 2, 3, 4\} \cup \{1, 3, 5, 7\} = \{1, 2, 3, 4, 5, 7\}$

Note that we do not list the same element twice in set notation.
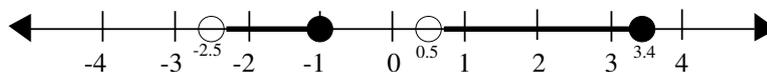
Example 2: $(-2.5, -1] \cup (0.5, 3.4]$

Figure 1.3: $(-2.5, 1] \cup (0.5, 3.4]$

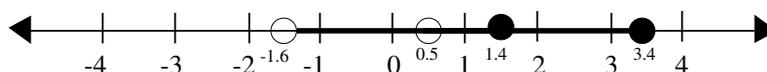**Example 3:** $(-1.6, 1.4] \cup (0.5, 3.4] = (-1.6, 3.4]$

Figure 1.4: $(-1.6, 3.4]$

Numerical Expressions

There are certain things to keep in mind when working with numerical expressions. When adding or multiplying two numbers the order is not important. For example, $5 + 3 = 8 = 3 + 5$. In general, for all real numbers,

$$a + b = b + a \quad \text{and} \quad ab = ba \quad \textbf{(commutativity)}$$

The same is not true for subtraction or division, since $5 - 3 \neq 3 - 5$ and $5/3 \neq 3/5$. Moreover, when we add or multiply numbers we do not need parentheses to specify the order in which the operation should be performed, since

$$(a + b) + c = a + (b + c) \quad \text{and} \quad a(bc) = (ab)c \quad \textbf{(associativity)}$$

But, again, this is not true for subtraction or division, since $5 - (3 - 2) = 5 - 1 = 4$, while $(5 - 3) - 2 = 2 - 2 = 0$.

Thus we need parentheses when we perform subtraction or when we deal with more than one operation, since $(5 \cdot 3) + 2 = 15 + 2 = 17$, while $5 \cdot (3 + 2) = 5 \cdot 5 = 25$. This creates an overabundance of parentheses, so we agree to perform multiplications

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and divisions first and then additions and subtractions; moreover, we perform like operations from left to right. So, $5 \cdot 3 + 2$ means $15 + 2 = 17$, while $5 - 3 - 2$ means $2 - 2 = 0$.

It is easy to observe that $5 \cdot (3 + 2) = 5 \cdot 3 + 5 \cdot 2$, and this a general property of numbers, which also holds if we have subtraction instead of addition:

$$a(b + c) = ab + ac \text{ and } (b + c)a = ba + ca \quad \textbf{(distributivity)}$$
$$a(b - c) = ab - ac \text{ and } (b - c)a = ba - ca \quad \textbf{(distributivity)}$$

If we take $a = -1$, we get $-(b + c) = -b - c$ and $-(b - c) = -b + c$, giving us a way of getting rid of parentheses when we have a minus sign in front.

Example 4:

$$2 - (5 - 1) - 3(-7 + 2) + 4(10 - 3) = 2 - 4 - 3(-5) + 4 \cdot 7 = 2 - 4 + 15 + 28 = 41$$

OR

$$2 - (5 - 1) - 3(-7 + 2) + 4(10 - 3) = 2 - 5 + 1 + 21 - 6 + 40 - 12 = 41$$

Absolute Values

Since a real number x is a point on the real line, we can consider its distance from the origin, which we call the **absolute value** of x and denote by $|x|$.
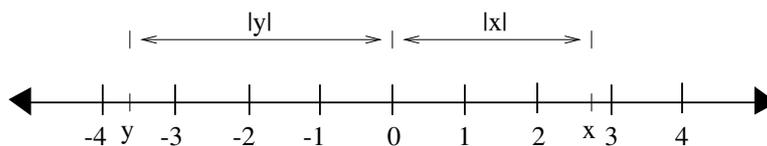
Figure 1.5: The absolute value is the distance from the origin

If the number x is positive, then $|x|$ is equal to the number itself ($|2| = 2$). If the number is negative, say $x = -2$, its distance $|x|$ from the origin is 2. Note that in this case the absolute value $|x| = 2$ is the negative of $x = -2$, namely $|x| = |-2| = 2 = -(-2) = -x$; so if x is negative, $|x| = -x$. The last equality is a bit awkward since $|x|$ is never negative, but we need the minus sign in front of x, since x has another negative sign "in"

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it; this way the two negative signs cancel out and we get a positive outcome. So we can define the absolute value of a number x as:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Here are some properties of the absolute value that hold for all real numbers:

Rule	Example
$\lvert -a \rvert = \lvert a \rvert$	$\lvert -2 \rvert = \lvert 2 \rvert$, because they both equal 2.
$\lvert ab \rvert = \lvert a \rvert \cdot \lvert b \rvert$	$\lvert (-2)3 \rvert = \lvert -2 \rvert \cdot \lvert 3 \rvert$, because both equal 6.
$\left\lvert \dfrac{a}{b} \right\rvert = \dfrac{\lvert a \rvert}{\lvert b \rvert}$	$\left\lvert \dfrac{2}{-3} \right\rvert = \dfrac{\lvert 2 \rvert}{\lvert -3 \rvert}$, since they both equal $\dfrac{2}{3}$.
$\lvert a + b \rvert \leq \lvert a \rvert + \lvert b \rvert$	$\lvert 4 + (-3) \rvert \leq \lvert 4 \rvert + \lvert -3 \rvert$, since $1 \leq 7$.

Example 5:

(a) $\lvert 1 - \sqrt{2} \rvert = -(1 - \sqrt{2})$, since $1 - \sqrt{2}$ is negative.

(b) $\left\lvert \dfrac{2 - \sqrt{2}}{\sqrt{2} - 2} \right\rvert = \dfrac{\lvert 2 - \sqrt{2} \rvert}{\lvert \sqrt{2} - 2 \rvert} = \dfrac{\lvert 2 - \sqrt{2} \rvert}{\lvert -(\sqrt{2} - 2) \rvert} = \dfrac{\lvert 2 - \sqrt{2} \rvert}{\lvert 2 - \sqrt{2} \rvert} = 1.$

Example 6: If $a \geq 2$, find $\lvert a - 1 \rvert$.

Since $a \geq 2$, we get $a \geq 1$. So $a - 1 \geq 0$, namely $a - 1$ is non-negative. Thus, $\lvert a - 1 \rvert = a - 1$.

Example 7: $\lvert x^2 + 1 \rvert = x^2 + 1$, since $x^2 + 1$ is positive for all values of x.

Example 8: If $x < -3$, calculate $\left\lvert 5\lvert x \rvert + 2 \right\rvert$.

Since $5\lvert x \rvert + 2$ is positive, $\left\lvert 5\lvert x \rvert + 2 \right\rvert = 5\lvert x \rvert + 2$. Now, since x is negative, $\lvert x \rvert = -x$; so, $5\lvert x \rvert + 2 = 5(-x) + 2 = -5x + 2$.

Example 9: Find all values of x such that $|x| = 3$.

Recall that $|x| = 3$ is the distance from x to the origin, so x can be only -3 or 3, namely $x = \pm 3$.

Example 10: Find all values of x such that $|2x - 5| = 7$.

Recall that $|2x - 5| = 7$ is the distance from $2x - 5$ to the origin; so $2x - 5$ can be only -7 or 7, namely $2x - 5 = \pm 7$. So, $2x - 5 = -7$ or $2x - 5 = 7$, namely $2x = -2$ or $2x = 12$. So $x = -1$ or $x = 6$.

Example 11: The set $\left\{ x \ \middle| \ |x| < 2 \right\}$ contains all numbers on the real line that are less than 2 units away from the origin.

Figure 1.6: The set $\left\{ x \ \middle| \ |x| < 2 \right\}$.
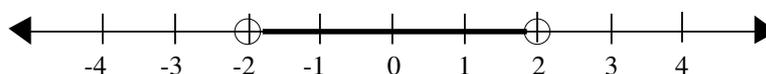
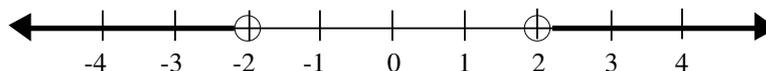**Example 12:** The set $\left\{ x \ \middle| \ |x| > 2 \right\}$ contains all numbers on the real line that are more than 2 units away from the origin.
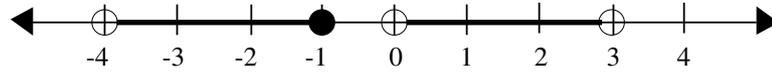

Figure 1.7: The set $\left\{ x \ \middle| \ |x| > 2 \right\}$.

Exercises

1. Rewrite $[-1, 3)$ from interval notation to set notation and draw your answer on the real number line.

2. Write the set $\{x | -3 < x \le 0\}$ in interval notation and draw your answer on the real number line.

3. Write down the following set of numbers in both interval and set notation.

4. Write the interval $[-2, 2]$ in set notation using absolute values. (See Examples 11 and 12.)

5. Find all values of x such that $|3 - x| = 2$.

6. Solve $|7x - 14| = 0$ for x.

7. Solve $|2x + 3| = 1$ for x.

8. Draw on the real number line the set of x-values for which each of the following is true. Use your drawing to write the set in interval notation.

(a) $|x| < 5$

(b) $|x| \geq 8$

(c) $|x - 2| < 6$

(d) $|x + 1| \geq 4$

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Exponents and Radicals

Exponents

If a and n are numbers, what do we mean when we write the expression a^n? Let's answer this question by beginning with the simplest cases where our intuition serves as a natural guide.

First, if n is a positive integer (that is, n could be 1,2,3, etc.), a^n just means that we multiply a by itself n times.

Example 1:

(a) $3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$

(b) $\left(\dfrac{1}{x}\right)^3 = \dfrac{1}{x} \cdot \dfrac{1}{x} \cdot \dfrac{1}{x} = \dfrac{1}{x^3}$

(c) $(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32$

Rules for Exponents

If this is what we mean by a^n when n is a positive integer, what rules should exponents obey?

Let us consider what $a^m \cdot a^n$ should be, when both m and n are positive integers. This is just a multiplied by itself m times, multiplied by a, n more times. Thus we have a times itself $m + n$ times. This is just a^{m+n}, and we have the following rule:

$$a^m \cdot a^n = a^{m+n} \tag{1}$$

Example 2: We have

$$5^3 \cdot 5^2 = (5 \cdot 5 \cdot 5)(5 \cdot 5) = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 5^5 = 5^{3+2}$$

This also tells us how we should define the expression a^0: if rule (1) is to hold for all numbers, we should have that $a^n = a^{n+0} = a^n \cdot a^0$, so that a^0 has to be 1 whenever $a \neq 0$. So let us make this definition for all numbers a which are not 0:

$$a^0 = 1 \text{ when } a \neq 0 \tag{2}$$

What other rules should exponents obey?

Consider what happens when we raise a power to a power, where both powers are integers:

$$(a^m)^n = (a \cdot a \cdot ... \cdot a) \cdot ... \cdot (a \cdot a \cdot ... \cdot a)$$

There are n groups of a's and each group contains m a's. Therefore we have mn a's total, and we now have the rule

$$(a^m)^n = a^{mn} \tag{3}$$

Example 3: We have

$$(4^2)^3 = (4 \cdot 4)^3 = (4 \cdot 4)(4 \cdot 4)(4 \cdot 4) = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^6 = 4^{2 \cdot 3}$$

Let's suppose that we want these nice rules to be obeyed no matter what numbers we choose as our exponents, and see how things have to work out.

If n is a positive integer and $a \neq 0$, we must have that $1 = a^0 = a^{n-n} = a^n \cdot a^{-n}$, so that when we divide both sides by a^n, we get that $\dfrac{1}{a^n} = a^{-n}$. Let us take this as our definition of a^{-n}, so that we can define powers with negative integers as exponents:

$$a^{-n} = \frac{1}{a^n} \tag{4}$$

Example 4:

(a) $x^{-3} = \dfrac{1}{x^3} = \dfrac{1}{x \cdot x \cdot x}$

(b) $13^{-2} = \dfrac{1}{13^2} = \dfrac{1}{(13) \cdot (13)} = \dfrac{1}{169}$

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Radicals

Suppose again that n is a positive integer. Since $a^1 = a$, we have that $a = a^1 = a^{n \cdot 1/n} = (a^{1/n})^n$ because we want rule (3) above to be true for all exponents. But this just says $a^{1/n}$ is the number that must be raised to the power n to get the number a. You may recall that we have a special name for this number: we call it the nth root of a, and we can write it using **radical notation** as $\sqrt[n]{a}$. (Since the square root occurs so frequently, we write it more simply: $a^{1/2} = \sqrt{a}$.) The work we just did shows the following formula:

$$a^{1/n} = \sqrt[n]{a}$$

We have just said that $\sqrt[n]{a}$ is the number that, when raised to the power n, gives the number a. When n is an even integer, there are generally *two* numbers that qualify for this definition. For example, 2 and -2, when squared, both give the number 4:

$$(2)^2 = 4 = (-2)^2$$

In such cases, we say that $\sqrt[n]{a}$ is the *nonnegative root*. So $\sqrt{4} = 2$, not -2, even though 2 and -2 are both square roots of 4. To represent both roots, we use the plus-or-minus sign:

$$\pm\sqrt{4} = \{-2, 2\}$$

Example 5:

(a) $\sqrt{169} = 169^{1/2} = 13$

(b) $2\sqrt[3]{8} = 2 \cdot 8^{1/3} = 2 \cdot 2 = 4$

More Rules for Exponents

You should get used to converting radicals to exponents and vice versa, since you will have to do this in many different kinds of problems.

Knowing how to define $a^{1/n}$ tells us how to define powers where our exponent is a positive rational number, that is, if our exponent is $\dfrac{m}{n}$ for two positive integers m and n. We have the formula

$$a^{m/n} = a^{m \cdot 1/n} = (a^m)^{1/n} = (a^{1/n})^m \tag{5}$$

again because rule (3) must still be obeyed.

Example 6:

$$32^{3/5} = (32^{1/5})^3 = 2^3 = 8$$

Finally, if we have a negative rational exponent $-\dfrac{m}{n}$ for positive integers m and n, we have $a^{-\frac{m}{n}} = \dfrac{1}{a^{\frac{m}{n}}}$ as was the case with negative integers.

There are two more details of which we must take note. Let n be a positive integer. If a and b are two numbers, it doesn't matter in what order we multiply a and b, so we have that

$$(ab)^n = (ab) \cdot (ab) \cdot ... \cdot (ab) = a \cdot a \cdot ... \cdot a \cdot b \cdot b \cdot ... \cdot b = a^n \cdot b^n$$

giving us that

$$(ab)^n = a^n \cdot b^n \tag{6a}$$

In fact, let us define this to be the case for *all* numbers n, so that this is a general rule for exponents.

Finally, we consider the expression $\left(\dfrac{a}{b}\right)^n$. Since $\dfrac{1}{b} = b^{-1}$, we have that $\dfrac{a}{b} = a \cdot \dfrac{1}{b} = a \cdot b^{-1}$. So we know that

$$\left(\frac{a}{b}\right)^n = (ab^{-1})^n = a^n \cdot (b^{-1})^n = a^n \cdot b^{-n} = \frac{a^n}{b^n}$$

for any number n that we may choose. This gives us the following formula:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \tag{6b}$$

Example 7:

(a) $(5x)^3 = 5^3 \cdot x^3 = 125x^3$

(b) $(3cd)^4 = 3^4 \cdot c^4 \cdot d^4 = 81 \cdot c^4 \cdot d^4 = 81c^4d^4$

(c) $\left(\dfrac{2x}{3}\right)^3 = \dfrac{(2x)^3}{3^3} = \dfrac{2^3 \cdot x^3}{27} = \dfrac{8x^3}{27}$

Let's summarize the rules that we've found for exponents. For any numbers a, x, and y, and any positive integer n, the following formulas are true:

$$a^0 = 1 \text{ when } a \neq 0$$

$$a^{x+y} = a^x \cdot a^y$$

$$a^{-x} = \frac{1}{a^x}$$

$$a^{1/n} = \sqrt[n]{a}$$

$$(a^x)^y = a^{xy}$$

$$a^{x/y} = (a^x)^{1/y} = (a^{1/y})^x$$

$$(ab)^x = a^x \cdot b^x$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Getting Rid of Radicals

Let's suppose that we are given the expression

$$4 + \sqrt{3x}$$

and we want to get rid of the radical. It is in this situation that we want to make use of the **conjugate** of the given expression:

$$4 - \sqrt{3x}$$

When we multiply these two expressions together, we obtain the following formula:

$$(4 + \sqrt{3x})(4 - \sqrt{3x}) = 16 + 4\sqrt{3x} - 4\sqrt{3x} - (\sqrt{3x})^2 = 16 - 3x$$

In general, the conjugate of an expression $a + b\sqrt{c}$ is given by the formula $a - b\sqrt{c}$, and just as in the example above, multiplying the first expression by its conjugate gets rid of the radical:

$$(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 + ab\sqrt{c} - ab\sqrt{c} - (b\sqrt{c})^2 = a^2 - b^2 c$$

Note that when we multiply an expression by its conjugate, the result is a new expression (without a radical) whose value is *different* from that of the expression with which we started.

Exercises

In problems 1-18, simplify each of the following expressions as much as possible, by writing them with as few and as simple exponents as you can.

1. $\dfrac{x^{7/3}}{x^{-2}}$

2. $\sqrt[4]{15} \cdot 15^{-1/3}$

3. $2^{3/2} 2^{7/2}$

4. $\sqrt[3]{x^{-2}}\sqrt{4x^5}$

5. $(2x)^4$

6. $\dfrac{16^{5/8} 16^{1/2}}{16^{7/8}}$

7. $8^{2/3}$

8. $\sqrt[3]{x^{3a+3b}}$

9. $1000^0 + 1^{1000}$

10. $\dfrac{(r^6)^n}{\sqrt[5]{r^{10n}}}$

11. $(7^{1/2})^4$

12. $\sqrt[3]{\dfrac{-8}{27}}$

13. $\dfrac{6^{2\pi} \cdot 6^{-\pi}}{6^\pi}$

14. $\dfrac{(xyz)^3}{xy^2 z^3}$

15. $2^0 \cdot 2^1 \cdot 2^2 \cdot 2^3 \cdot 2^4$

16. $((2^2)^2)^2$

17. $(x^y)^z (2^y)^{-z}$

18. $(6^0)^{1000}$

19. $\left(\dfrac{2xy^3}{4xy^2}\right)^3$

20. $(2xy^{-2}z)^{-3}(x^{-1}yz^{-2})$

21. $\left[(x^2y)^3(x^0y)^2\right]^4$

22. $\left(\dfrac{3xy^2}{2z}\right)^3 \left(\dfrac{4x^2z}{y^4}\right)^2$

23. $\left(-2^2 x^{2/3} y^{-1/2} z^{-2/3}\right)\left(x^{1/2} y^{3/2} z^{-1}\right)$

24. $\dfrac{\left(ab^2c\right)^{-3}}{\left(a^{-2}bc^2\right)^{-2}}$

25. $\left[\dfrac{\left(c^{-2}ab^{-4}\right)^2}{\left(2a^{-2}b^2\right)^3}\right]^{-2}$

In problems 26-29, find the conjugate of the given expression, and write out the expression which results from multiplying the given expression by its conjugate.

26. $3\pi + \sqrt{17x}$

27. $x - 5\sqrt{6}$

28. $-2 + x\sqrt{3+y}$

29. $\dfrac{1}{4} - \sqrt{\dfrac{2}{5}}$

Polynomials and Factoring

Polynomials

A **polynomial** in the variable x is an algebraic expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where a_n, a_{n-1}, ..., a_1, and a_0 are real numbers called the **coefficients** of the polynomial, $a_n \neq 0$, and n is a nonnegative integer called the **degree** of the polynomial.

A polynomial with degree 1 is called **linear** or **first degree**. A polynomial with degree 2 is called **quadratic** or **second degree**. A polynomial with degree 3 is called **cubic** or **third degree**. A polynomial with only one term is called a **monomial**, and a polynomial with exactly two terms is called a **binomial**.

Example 1:

(a) $4x^2 + x - 1$ is a quadratic polynomial.

(b) $-\frac{4}{3}x + 21$ is a linear polynomial.

(c) 7 is a monomial of degree 0.

(d) $x^{101} - 1$ is a binomial of degree 101.

Multiplying Polynomials

The **distributive law** of multiplication over addition and subtraction says that for any real numbers a, b, and c,

$$a(b + c) = ab + ac, \text{ and } a(b - c) = ab - ac.$$

When multiplying two polynomials, the distributive law means that every term in the first polynomial must be multiplied by every term in the second:

$$(x + 2)(2x^2 - 4x + 1) = (x)(2x^2) - (x)(4x) + (x)(1) + (2)(2x^2) - (2)(4x) + (2)(1).$$

We can simplify this expression by multiplying and collecting like powers of x:

$$\begin{aligned}
(x+2)(2x^2 - 4x + 1) &= 2x^3 - 4x^2 + x + 4x^2 - 8x + 2 \\
&= (2x^3) + (-4x^2 + 4x^2) + (x - 8x) + (2) \\
&= 2x^3 + (-4 + 4)x^2 + (1 - 8)x + 2 \\
&= 2x^3 - 7x + 2.
\end{aligned}$$

When multiplying two binomials, students sometimes find it helpful to remember the acronym FOIL, which stands for "First Outer Inner Last."

Example 2:

$$\begin{aligned}
(4x^4 + 1)(x^2 + 3x) &= \text{product of First terms } + \text{ product of Outer terms} \\
&\quad + \text{ product of Inner terms } + \text{ product of Last terms} \\
&= (4x^4)(x^2) + (4x^4)(3x) + (1)(x^2) + (1)(3x) \\
&= 4x^6 + 12x^5 + x^2 + 3x
\end{aligned}$$

Factoring Polynomials

Factoring a given polynomial means finding two or more polynomials of smaller degree whose product is the given polynomial. It is like "undoing" multiplication.

For example, $4x^2 - 4x - 3$ factors as $(2x + 1)(2x - 3)$ because

$$(2x + 1)(2x - 3) = 4x^2 - 6x + 2x - 3 = 4x^2 - 4x - 3.$$

HOW DO WE FACTOR A POLYNOMIAL?

1. Try finding a **common factor**, a term that divides evenly into each term of the polynomial.

Example 3: Find a common factor of $2x^3 + 4x^2 - 14x$.

Since $2x$ divides every term evenly, we can factor out $2x$ to get

$$2x^3 + 4x^2 - 14x = 2x(x^2 + 2x - 7).$$

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2. Try using one of the following **factoring formulas**:

$$
\begin{aligned}
(1) \quad & a^2 - b^2 = (a - b)(a + b) \\
(2) \quad & a^2 + 2ab + b^2 = (a + b)^2 \\
(3) \quad & a^2 - 2ab + b^2 = (a - b)^2 \\
(4) \quad & a^3 - b^3 = (a - b)(a^2 + ab + b^2) \\
(5) \quad & a^3 + b^3 = (a + b)(a^2 - ab + b^2)
\end{aligned}
$$

Example 4:

$$
\begin{aligned}
4x^4 - 9x^2 &= x^2(4x^2 - 9) && \text{Common Factor} \\
&= x^2((2x)^2 - 3^2) \\
&= x^2(2x - 3)(2x + 3) && \text{Formula (1)}
\end{aligned}
$$

Example 5:

$$
\begin{aligned}
9x^2 + 48x + 64 &= (3x)^2 + 2(3x)(8) + 8^2 \\
&= (3x + 8)^2 && \text{Formula (2)}
\end{aligned}
$$

3. If the polynomial in question is quadratic, then the following formula may be useful:

$$
x^2 + (a + b)x + ab = (x + a)(x + b).
$$

Given a polynomial of the form $x^2 + cx + d$, to use this formula, try to find two numbers a and b whose sum is c and whose product is d.

Example 6: Factor $x^2 + 13x + 22$.

Since $11 + 2 = 13$ and $11 \cdot 2 = 22$, we can find the factors:

$$
x^2 + 13x + 22 = (x + 11)(x + 2).
$$

(4) If the polynomial has four terms, it may be possible to **factor by grouping**.

Example 7: Factor $2x^3 - 8x^2 + 3x - 12$.

Note that the first two terms have a common factor, as do the last two terms. Group terms with common factors and pull out those factors:

$$2x^3 - 8x^2 + 3x - 12 = (2x^3 - 8x^2) + (3x - 12)$$
$$= 2x^2(x - 4) + 3(x - 4)$$

Now we have two terms, each of which has a factor of $(x - 4)$. Pull out that common factor:

$$2x^3 - 8x^2 + 3x + 12 = 2x^2(x - 4) + 3(x - 4)$$
$$= (x - 4)(2x^2 + 3)$$

Roots of Polynomials

The **roots** of a polynomial $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ are those numbers r for which

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0.$$

Example 8: $x = 2$ is a root of the polynomial $x^2 + x - 6$ because $2^2 + 2 - 6 = 0$.

A polynomial can have only as many distinct roots as its degree. (It may have fewer, but it cannot have more.) If $x = r$ is a root of a given polynomial, then $x - r$ is a factor of that polynomial.

HOW DO WE FIND THE ROOTS OF A POLYNOMIAL?

(1) Try factoring the polynomial. Since a product equals zero if and only if any of its factors equals zero, the roots of a polynomial are the roots of its factors.

Example 9: What are the roots of $x^2 - 6x - 40$?

In other words, what x-values make $x^2 - 6x - 40 = 0$?

$$x^2 - 6x - 40 = 0$$
$$(x - 10)(x + 4) = 0$$
$$x - 10 = 0 \quad \text{or} \quad x + 4 = 0$$
$$x = 10 \quad \text{or} \quad x = -4$$

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Thus $x^2 - 6x - 40$ has roots $x = -4$, 10.

Example 10: What are the roots of $4x^4 - 9x^2$?

$$4x^4 - 9x^2 = 0$$
$$x^2(2x - 3)(2x + 3) = 0$$
$$x^2 = 0 \quad \text{or} \quad 2x - 3 = 0 \quad \text{or} \quad 2x + 3 = 0$$
$$x = 0 \quad \text{or} \quad x = \frac{3}{2} \quad \text{or} \quad x = -\frac{3}{2}$$

Thus $4x^4 - 9x^2$ has roots $x = 0$, $\frac{3}{2}$, $-\frac{3}{2}$.

(2) If the polynomial is quadratic and of the form

$$\boxed{ax^2 + bx + c}$$

then its roots are given by the **quadratic formula**:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The quadratic formula will always find the roots of a quadratic polynomial. However, factoring is sometimes faster.

Example 11: What are the roots of $2x^2 - 4x + 1$?

Here $a = 2$, $b = -4$, and $c = 1$, so the quadratic formula gives roots

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(1)}}{2(2)} = \frac{2 \pm \sqrt{2}}{2}.$$

Thus $2x^2 - 4x + 1$ has roots $x = \frac{2 + \sqrt{2}}{2}$ and $x = \frac{2 - \sqrt{2}}{2}$.

Example 12: What are the roots of $x^2 - 4x + 9$?

In this case, the quadratic formula gives

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(9)}}{2(1)} = \frac{4 \pm \sqrt{-20}}{2}.$$

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Whenever a root involves the square root of a negative number, we call this root **imaginary**. Imaginary roots are not real numbers and do not count as real roots. It is assumed throughout most calculus courses that roots must be real numbers. When this assumption is made, imaginary roots do not count as roots. Thus to answer the question, "What are the roots of $x^2 - 4x + 9$?", we would say that $x^2 - 4x + 9$ has no real roots since its only roots are imaginary.

Since this polynomial is quadratic and has no real roots, we call it an **irreducible quadratic**.

Advanced Root-Finding Techniques

The three techniques described above will be enough to find the find the roots of many of the polynomials you will encounter. However, to find the roots of some polynomials, especially polynomials of degree 3 or higher, you will need more advanced techniques, such as the following:

1. The **Rational Root Theorem** says that if the coefficients of a polynomial $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ are integers, then any rational root (that is, any root that can be expressed as a fraction of integers) has the form $\pm \frac{p}{q}$ where p and q are positive integers, p is a divisor of a_0, and q is a divisor of a_n.

Example 13: Find all rational roots of $3x^3 + x^2 - 12x - 4$.

Here $n = 3$, $a_n = 3$, and $a_0 = -4$. Since p is a divisor of $a_0 = -4$, p could be 1, 2, or 4. Since q is a divisor of $a_n = 3$, q could be 1 or 3. Hence any rational root of the polynomial must be one of the following:

$$\pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$$

To determine which of these actually are roots, substitute each one into the polynomial:

x	$3x^3 + x^2 - 12x - 4$
1	-12
-1	6
2	0
-2	0
4	156
-4	-132
$\frac{1}{3}$	$-\frac{70}{9}$
$-\frac{1}{3}$	0

Since the polynomial is cubic, it has at most three roots. We have found three roots ($x = 2$, -2, and $-\frac{1}{3}$), so we have found *all* the roots of the polynomial.

2. If you can find one root of a polynomial (perhaps by applying the Rational Root Theorem as we did in the last example), there are two methods of attempting to find the remaining roots. One of these methods is called **polynomial long division**.

Example 14: Find all roots of $x^4 - x^3 - 7x^2 - 14x - 24$.

The Rational Root Theorem tells us that all rational roots must be of the form $\pm\frac{p}{q}$, where p and q are positive integers, p is a divisor of $a_0 = -24$, and q is a divisor of $a_n = 1$. Hence the candidates for rational roots are

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24.$$

Since $(4)^4 - (4)^3 - 7(4)^2 - 14(4) - 24 = 0$, $x = 4$ is a root and so $x - 4$ is a factor. Use polynomial long division to divide the polynomial by $x - 4$:

$$
\begin{array}{r|lllll}
& x^3 & +3x^2 & +5x & +6 \\
\hline
x-4 & x^4 & -x^3 & -7x^2 & -14x & -24 \\
& -(x^4 & -4x^3) \\
\hline
& & 3x^3 & -7x^2 & -14x & -24 \\
& & -(3x^3 & -12x^2) \\
\hline
& & & 5x^2 & -14x & -24 \\
& & & -(5x^2 & -20x) \\
\hline
& & & & 6x & -24 \\
& & & & -(6x & -24) \\
\hline
& & & & & 0
\end{array}
$$

(Since we knew $x - 4$ was a factor of our polynomial, we expected to get 0 as our final remainder.)

Thus $x^4 - x^3 - 7x^2 - 14x - 24 = (x-4)(x^3+3x^2+5x+6)$. To factor x^3+3x^2+5x+6, we apply the Rational Root Theorem and get candidate rational roots $\pm 1, \pm 2, \pm 3, \pm 6$. Since $(-2)^3 + 3(-2)^2 + 5(-2) + 6 = 0$, $x = -2$ is a root and so $x - (-2) = x + 2$ is a factor. Use polynomial long division again:

$$
\begin{array}{r|llll}
& x^2 & +x & +3 \\
\hline
x+2 & x^3 & +3x^2 & +5x & +6 \\
& -(x^3 & +2x^2) \\
\hline
& & x^2 & +5x & +6 \\
& & -(x^2 & +2x) \\
\hline
& & & 3x & +6 \\
& & & -(3x) & +6) \\
\hline
& & & & 0
\end{array}
$$

(As before, we expected to get 0 as our final remainder since we knew that $x + 2$ was a factor of $x^3 + 3x^2 + 5x + 6$.)

Hence $x^4 - x^3 - 7x^2 - 14x - 24 = (x - 4)(x + 2)(x^2 + x + 3)$. To find the roots of $x^2 + x + 3$, use the quadratic formula:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(3)}}{2(1)} = \frac{-1 \pm \sqrt{-11}}{2}$$

These are imaginary roots, and so the real roots of the original polynomial are $x = 4$ and $x = -2$.

3. Given one root of a polynomial, the other method for finding the remaining roots is a process called **synthetic division**. While polynomial long division lets you divide any polynomial into any other polynomial, synthetic division is only possible if the divisor is a polynomial of the form $x - r$. (Keep in mind that r could be negative, so $x + 2 = x - (-2)$ is such a polynomial.)

To use synthetic division to divide $x - r$ into $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, begin with the following diagram:

$$\begin{array}{r|cccc|c} r & a_n & a_{n-1} & \cdots & a_1 & a_0 \\ & & & & & \\ \hline & & & & & \\ \end{array}$$

If a power of x is missing from the polynomial, make sure to place a 0 in top row of the corresponding column in the diagram. For example, to divide $x + 2$ into $x^4 - x^2 + 1$, begin with

$$\begin{array}{r|cccc|c} -2 & 1 & 0 & -1 & 0 & 1 \\ & & & & & \\ \hline & & & & & \\ \end{array}$$

Next, write a_n in the bottom row of the column with a_n in the top row. Then, for each column from left to right,

(a) multiply the value in the bottom row of the previous column by r,

(b) write this value in the middle row,

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(c) add this value to the value in the top row, and

(d) write the result of this addition in the bottom row.

Eventually you will get a number in the lower righthand corner of the diagram. If this number is not 0, then $x - r$ is not a factor of the polynomial and so r is not a root. If this number is 0, then $x - r$ is a factor of the polynomial, r is a root of the polynomial, and the numbers in the bottom row to the left of the vertical line tell you the coefficients of the other factor, the polynomial $\dfrac{a_n x^n + \ldots + a_0}{x - r}$.

Example 15: Find all roots of $x^3 - 4x^2 - 2x + 3$.

The Rational Root Theorem gives us candidate rational roots ± 1 and ± 3. Use synthetic division to see if 1 is a root:

$$
\begin{array}{r|rrr|r}
1 & 1 & -4 & -2 & 3 \\
  &   & 1 & -3 & -5 \\
\hline
  & 1 & -3 & -5 & -2
\end{array}
$$

Since $-2 \neq 0$, $x = 1$ is not a root. Try $x = -1$:

$$
\begin{array}{r|rrr|r}
-1 & 1 & -4 & -2 & 3 \\
   &   & -1 & 5 & -3 \\
\hline
   & 1 & -5 & 3 & 0
\end{array}
$$

Since 0 appears in the lower righthand corner, $x = -1$ is a root and

$$x^3 - 4x^2 - 2x + 3 = (x + 1)(x^2 - 5x + 3).$$

Note how the numbers in the bottom row of the diagram $(1, -5, \text{ and } 3)$ give us the coefficients of the other factor.

To find the roots of $x^2 - 5x + 3$, use the quadratic formula:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)} = \frac{5 \pm \sqrt{13}}{2}$$

Hence the roots of the original polynomial are $x = -1$, $\dfrac{5 + \sqrt{13}}{2}$, and $\dfrac{5 - \sqrt{13}}{2}$.

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Exercises

Determine the degree of the following polynomials.

1. $7x^2 - 1$
2. $1 - x$
3. 14
4. $ax^3 + bx^2 + cx + d$

Simplify the following expressions by multiplying and collecting like powers of x.

5. $2x(x^3 - 1)$
6. $(4x^4 + 2x^2 - 7)x^2$
7. $(5x - 1)(x + 6)$
8. $(x^2 - 1)(x^2 + 1)$
9. $(x + 2)^2$
10. $(x - 1)^2$
11. $(2x^2 - 1)(1 - x - 3x^2)$
12. $(-x^2 + 2x - 3)(14x^3 + 1)$
13. $(ax + b)(cx + d)$
14. $(ax + b)^3$
15. $x(x + 1)(x + 2)$
16. $(x - 2)(x + 3)^2$

Factor completely the following polynomials.

17. $x^2 + x$
18. $3x^4 + 9x^3$
19. $2x^3 + 4x^2 + 2x$
20. $-x^2 + 6x - 9$
21. $16x^2 - 25$
22. $49x^2 - 81x^4$
23. $8 - x^3$
24. $27x^3 + 1$
25. $x^2 + 12x + 27$
26. $2x^2 + 8x + 6$
27. $x^2 - 2x - 8$
28. $x^2 + 4x - 21$
29. $x^3 + x^2 - x - 1$
30. $16x^4 - 16x^3 + 2x - 2$
31. $x^4 + 2x^3 - x - 2$
32. $x^4 - 8x^2 + 16$

Find all real roots of the following polynomials.

33. $x^2 + 10x - 11$ 34. $x^2 - 4x + 4$

35. $x^2 - 9$ 36. $x^2 + 25x + 100$

37. $x^2 - x - 1$ 38. $x^3 + 2x^2 + 2x$

39. $-x^2 + 4x + 1$ 40. $2x^2 - 7x + 2$

41. $x^2 + 4x + 9$ 42. $3x^2 + x + 2$

Use the advanced root-finding techniques discussed in this section to find all real roots of the following polynomials.

43. $x^3 - 6x^2 + x - 6$ 44. $x^3 + x^2 - 24x + 36$

45. $2x^4 - 9x^3 - 2x^2 + 39x - 18$ 46. $x^4 - 4x^3 + 6x^2 - 4x + 1$

47. $2x^4 + 15x^3 + 25x^2 - 8x + 16$ 48. $4x^5 - 16x^4 + 17x^3 - 19x^2 + 13x - 3$

49. $x^4 - 7x^2 + 12$ 50. $x^4 - x^3 - 2x - 4$

51. $x^4 - 13x^2 - 12x$ 52. $3x^4 - 3x^3 - 6x - 12$

Algebraic Fractions

The simplest form of an algebraic fraction is a **rational expression,** which is a quotient of two polynomials. Examples of rational expressions are

$$\frac{2}{5ab}, \quad \frac{3xy + 4y^3}{7x}, \quad \text{and} \quad \frac{x+5}{8}.$$

Lowest Terms

A fraction is said to be in **lowest terms** if all common factors of the numerator and denominator have been canceled.

Example 1: Reduce the following fractions to lowest terms:

(a) $\dfrac{x^4 - 2x^3}{x^2 - 2x} = \dfrac{x^3(x-2)}{x(x-2)} = x^2$

(b) $\dfrac{2x^2 + x - 3}{4 - 5x + x^2} = \dfrac{(x-1)(2x+3)}{(1-x)(4-x)} = \dfrac{(x-1)(2x+3)}{-(x-1)(4-x)} = \dfrac{2x+3}{-(4-x)} = \dfrac{2x+3}{x-4}$

A WARNING ABOUT CANCELING:

$$\frac{3x + 7y}{x} \neq 3 + 7y$$

We cannot cancel the x's because x is not a factor of *every* term in the numerator and *every* term in the denominator. The fraction $\dfrac{3x + 7y}{x}$ is already in lowest terms.

Multiplying and Dividing

When multiplying fractions we use the following rule:

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$$

that is, we multiply numerators together and denominators together.

Example 2: Multiply and reduce to lowest terms:

(a) $\dfrac{x+2}{x+1} \cdot \dfrac{x}{x+1} = \dfrac{x(x+2)}{(x+1)^2}$

(b) $\dfrac{xy}{wz} \cdot \dfrac{w^2 z}{x^2 y^2} = \dfrac{xyw^2 z}{wzx^2 y^2} = \dfrac{w}{xy}$

(c) $\dfrac{2x-8}{x+2} \cdot \dfrac{x^2+4x+4}{x^2-16} = \dfrac{2(x-4)}{x+2} \cdot \dfrac{(x+2)^2}{(x+4)(x-4)} = \dfrac{2(x-4)(x+2)^2}{(x+2)(x+4)(x-4)}$

$= \dfrac{2(x+2)}{x+4}$

(d) $a\left(\dfrac{bc}{a^2}\right) = \dfrac{a}{1} \cdot \dfrac{bc}{a^2} = \dfrac{abc}{a^2} = \dfrac{bc}{a}$

In order to divide fractions, we use the following rule:

$$\frac{\left(\dfrac{P}{Q}\right)}{\left(\dfrac{R}{S}\right)} = \dfrac{P}{Q} \cdot \dfrac{S}{R}$$

that is, we multiply the numerator by the reciprocal of the expression in the denominator.

Example 3: Divide and reduce to lowest terms:

(a) $\dfrac{\dfrac{xy}{x+y}}{\dfrac{x^2 y}{(x+y)^3}} = \dfrac{xy}{x+y} \cdot \dfrac{(x+y)^3}{x^2 y} = \dfrac{xy(x+y)^3}{x^2 y(x+y)} = \dfrac{(x+y)^2}{x}$

(b) $\dfrac{\dfrac{xy-zx}{w}}{\dfrac{x}{w} \cdot y} = \dfrac{\dfrac{xy-zx}{w}}{\dfrac{xy}{w}} = \dfrac{xy-zx}{w} \cdot \dfrac{w}{xy} = \dfrac{x(y-z)}{w} \cdot \dfrac{w}{xy} = \dfrac{xw(y-z)}{xwy} = \dfrac{y-z}{y}$

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Adding and Subtracting

When adding or subtracting fractions, there are two cases to consider.

Case 1: Denominators are the same

When adding or subtracting fractions with the same denominators, we use the following rules:
$$\frac{P}{Q} + \frac{R}{Q} = \frac{P+R}{Q} \quad \text{or} \quad \frac{P}{Q} - \frac{R}{Q} = \frac{P-R}{Q}$$
that is, we add or subtract the numerators and keep the common denominator.

Example 4: Add or subtract and reduce to lowest terms:

(a) $\dfrac{a}{b+2} - \dfrac{a+b}{b+2} = \dfrac{a-(a+b)}{b+2} = \dfrac{a-a-b}{b+2} = \dfrac{-b}{b+2}$

(b) $\dfrac{4x^2 - 5xy}{x^2 - y^2} + \dfrac{2xy - y^2}{x^2 - y^2} = \dfrac{4x^2 - 3xy - y^2}{x^2 - y^2} = \dfrac{(4x+y)(x-y)}{(x+y)(x-y)} = \dfrac{4x+y}{x+y}$

Case 2: Denominators are different

When adding or subtracting fractions with different denominators, we first find a common denominator. There are many common denominators we can use, but the "best" one is usually the Least Common Denominator (LCD). To find the LCD of the given fractions, first factor the denominators. Then use each factor the greatest number of times it occurs in any one of the denominators.

Example 5: In order to calculate $\dfrac{a}{(a-b)^2} + \dfrac{a}{a-b}$, we use as a common denominator $(a-b)^2$ since $(a-b)$ is the *only* factor occurring in the denominators, and the greatest number of times it occurs in any one of the denominators is two.

We compute: $\dfrac{a}{(a-b)^2} + \dfrac{a}{a-b} = \dfrac{a}{(a-b)^2} + \dfrac{a(a-b)}{(a-b)^2} = \dfrac{a}{(a-b)^2} + \dfrac{a^2 - ab}{(a-b)^2}$

$= \dfrac{a + a^2 - ab}{(a-b)^2}.$

Notice what happens with the above example if we had used as our common denominator the product of the individual denominators instead of the LCD:

$$\frac{a}{(a-b)^2} + \frac{a}{a-b} = \frac{a(a-b)}{(a-b)^3} + \frac{a(a-b)^2}{(a-b)^3} = \frac{a(a-b) + a(a-b)^2}{(a-b)^3} = \frac{(a-b)[(a+a(a-b)]}{(a-b)^3}$$

$$= \frac{a + a(a-b)}{(a-b)^2} = \frac{a + a^2 - ab}{(a-b)^2}.$$

We get the same answer, but it takes more work!

Example 6: Add or subtract and reduce to lowest terms:

(a) $\dfrac{x}{x-2} - \dfrac{2}{x^2-4} = \dfrac{x}{x-2} - \dfrac{2}{(x-2)(x+2)} = \dfrac{x(x+2)}{(x-2)(x+2)} - \dfrac{2}{(x-2)(x+2)}$

$= \dfrac{x(x+2) - 2}{(x-2)(x+2)} = \dfrac{x^2 + 2x - 2}{(x-2)(x+2)}$

(b) $\dfrac{x}{y^2 z} + \dfrac{x}{yz^2} - \dfrac{x}{yz} = \dfrac{xz}{y^2 z^2} + \dfrac{xy}{y^2 z^2} - \dfrac{xyz}{y^2 z^2} = \dfrac{xz + xy - xyz}{y^2 z^2}$

(c) $\dfrac{a+1}{3a} + \dfrac{b-2}{5b} = \dfrac{5b(a+1)}{15ab} + \dfrac{3a(b-2)}{15ab} = \dfrac{5b(a+1) + 3a(b-2)}{15ab} = \dfrac{5ab + 5b + 3ab - 6a}{15ab}$

$= \dfrac{-6a + 8ab + 5b}{15ab}$

A WARNING ABOUT FRACTIONS:

$$\frac{x}{y+2} \neq \frac{x}{y} + \frac{x}{2}$$

If we add the fractions on the right hand side, we see that we get $\dfrac{xy + 2x}{2y}$, *not* $\dfrac{x}{y+2}$.

Other Algebraic Fractions

The techniques used to simplify rational expressions may also be used to simplify algebraic fractions in which the numerator and/or denominator themselves contain fractional

expressions. These types of fractions are called **complex fractions**. Two examples of complex fractions are

$$\frac{4x}{x - x^{-2}} \quad \text{and} \quad \frac{\dfrac{1}{\sqrt{a} + 4}}{\dfrac{2}{b} - \sqrt[3]{b}}.$$

We use the following guidelines to simplify complex fractions.

Simplification Guidelines for Complex Fractions

Method 1:

1. Simplify the numerator and the denominator separately to obtain a single fractional expression in each.

2. Treat the result as a division problem.

3. Simplify as usual.

Method 2:

1. Find the LCD of all denominators appearing in the entire fraction.

2. Multiply the entire fraction by 1, using $\dfrac{n}{n}$ where n is the LCD found in step 1.

3. Simplify as usual.

Example 7: Perform the given operations and simplify:

(a) $\dfrac{1 + \dfrac{1}{x}}{1 - \dfrac{1}{x^2}}$

Using Method 1:

$$\frac{1 + \dfrac{1}{x}}{1 - \dfrac{1}{x^2}} = \frac{\dfrac{x}{x} + \dfrac{1}{x}}{\dfrac{x^2}{x^2} - \dfrac{1}{x^2}} = \frac{\dfrac{x+1}{x}}{\dfrac{x^2-1}{x^2}} = \frac{x+1}{x} \cdot \frac{x^2}{x^2-1} = \frac{(x+1)x^2}{x(x+1)(x-1)} = \frac{x}{x-1}$$

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Using Method 2:

The expression $\dfrac{1 + \dfrac{1}{x}}{1 - \dfrac{1}{x^2}}$ has denominators x and x^2; the LCD is x^2.

So $\dfrac{1 + \dfrac{1}{x}}{1 - \dfrac{1}{x^2}} = \dfrac{1 + \dfrac{1}{x}}{1 - \dfrac{1}{x^2}} \cdot \dfrac{x^2}{x^2} = \dfrac{\left(1 + \dfrac{1}{x}\right) x^2}{\left(1 - \dfrac{1}{x^2}\right) x^2} = \dfrac{x^2 + x}{x^2 - 1} = \dfrac{x(x+1)}{(x+1)(x-1)} = \dfrac{x}{x-1}$

(b) $\dfrac{(x^2 + 1)^{\frac{1}{2}} - 2x^2(x^2 + 1)^{-\frac{1}{2}}}{1 - x^2}$

Using Method 1:

$$\frac{(x^2 + 1)^{\frac{1}{2}} - 2x^2(x^2 + 1)^{-\frac{1}{2}}}{1 - x^2} = \frac{(x^2 + 1)^{\frac{1}{2}} - \dfrac{2x^2}{(x^2 + 1)^{\frac{1}{2}}}}{1 - x^2} = \frac{\dfrac{x^2 + 1}{(x^2 + 1)^{\frac{1}{2}}} - \dfrac{2x^2}{(x^2 + 1)^{\frac{1}{2}}}}{1 - x^2}$$

$$= \frac{\dfrac{x^2 + 1 - 2x^2}{(x^2 + 1)^{\frac{1}{2}}}}{1 - x^2} = \frac{\dfrac{1 - x^2}{(x^2 + 1)^{\frac{1}{2}}}}{1 - x^2} = \frac{1 - x^2}{(x^2 + 1)^{\frac{1}{2}}} \cdot \frac{1}{1 - x^2} = \frac{1}{(x^2 + 1)^{\frac{1}{2}}}$$

Using Method 2:

First write $\dfrac{(x^2 + 1)^{\frac{1}{2}} - 2x^2(x^2 + 1)^{-\frac{1}{2}}}{1 - x^2} = \dfrac{(x^2 + 1)^{\frac{1}{2}} - \dfrac{2x^2}{(x^2 + 1)^{\frac{1}{2}}}}{1 - x^2}$.

This expression has the denominator $(x^2+1)^{\frac{1}{2}}$; the LCD is $(x^2+1)^{\frac{1}{2}}$.

$$\text{So } \frac{(x^2+1)^{\frac{1}{2}} - 2x^2(x^2+1)^{-\frac{1}{2}}}{1-x^2} = \frac{(x^2+1)^{\frac{1}{2}} - \dfrac{2x^2}{(x^2+1)^{\frac{1}{2}}}}{1-x^2} \cdot \frac{(x^2+1)^{\frac{1}{2}}}{(x^2+1)^{\frac{1}{2}}}$$

$$= \frac{\left[(x^2+1)^{\frac{1}{2}} - \dfrac{2x^2}{(x^2+1)^{\frac{1}{2}}}\right](x^2+1)^{\frac{1}{2}}}{(1-x^2)(x^2+1)^{\frac{1}{2}}} = \frac{\left[(x^2+1)^{\frac{1}{2}}(x^2+1)^{\frac{1}{2}} - \dfrac{2x^2(x^2+1)^{\frac{1}{2}}}{(x^2+1)^{\frac{1}{2}}}\right]}{(1-x^2)(x^2+1)^{\frac{1}{2}}}$$

$$= \frac{x^2+1-2x^2}{(1-x^2)(x^2+1)^{\frac{1}{2}}} = \frac{1-x^2}{(1-x^2)(x^2+1)^{\frac{1}{2}}} = \frac{1}{(x^2+1)^{\frac{1}{2}}}$$

It is a matter of personal taste as to which of the methods in the above examples one uses.

Rationalizing Fractions

When fractions have numerators or denominators containing sums or differences involving radicals, we may sometimes want to **rationalize** the numerator or denominator. Rationalizing the numerator means getting rid of any radicals in the numerator. Likewise, rationalizing the denominator means getting rid of any radicals in the denominator.

Example 8: Rationalize the numerator of $\dfrac{\sqrt{x}}{1+\sqrt{x}}$ and reduce to lowest terms.

In order to get rid of the radical $\sqrt{x}$ in the numerator, we multiply the given fraction by $\dfrac{\sqrt{x}}{\sqrt{x}}$ and compute:

$$\frac{\sqrt{x}}{1+\sqrt{x}} = \frac{\sqrt{x}}{1+\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{(\sqrt{x})^2}{(1+\sqrt{x})\sqrt{x}} = \frac{(\sqrt{x})^2}{\sqrt{x}+(\sqrt{x})^2} = \frac{x}{\sqrt{x}+x}.$$

Note that we must multiply numerator *and* denominator by $\sqrt{x}$ so that we don't change the value of the original fraction – we're only multiplying by the number 1!

When rationalizing the numerator or denominator of a fraction, we sometimes need to use the **conjugate** of a given expression. Recall that to obtain the conjugate of an two-term expression involving radicals, we change the sign between the terms. For example, the conjugate of $\sqrt{3} + \sqrt{x}$ is $\sqrt{3} - \sqrt{x}$, and the conjugate of $\sqrt{x+2} - 1$ is $\sqrt{x+2} + 1$.

Example 9: Rationalize the denominator of $\dfrac{\sqrt{x}}{1+\sqrt{x}}$ and reduce to lowest terms.

We make use of the conjugate of $1 + \sqrt{x}$, which is $1 - \sqrt{x}$. In order to get rid of the radical $\sqrt{x}$ in the denominator, we multiply the given fraction by $\dfrac{1-\sqrt{x}}{1-\sqrt{x}}$ and compute:

$$\frac{\sqrt{x}}{1+\sqrt{x}} = \frac{\sqrt{x}}{1+\sqrt{x}} \cdot \frac{1-\sqrt{x}}{1-\sqrt{x}} = \frac{\sqrt{x}(1-\sqrt{x})}{(1+\sqrt{x})(1-\sqrt{x})} = \frac{\sqrt{x} - (\sqrt{x})^2}{1^2 - (\sqrt{x})^2} = \frac{\sqrt{x} - x}{1 - x}.$$

Again, we must multiply numerator *and* denominator by $1 - \sqrt{x}$, so that we don't change the value of the original fraction.

Example 10: Rationalize the numerator of $\dfrac{\sqrt{x+h} - \sqrt{x}}{h}$ and reduce to lowest terms.

We make use of the conjugate of $\sqrt{x+h} - \sqrt{x}$, which is $\sqrt{x+h} + \sqrt{x}$. In order to get rid of the radicals $\sqrt{x+h}$ and $\sqrt{x}$ in the numerator, we multiply the given fraction by $\dfrac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$ and compute:

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} = \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

Note that in Examples 8, 9, and 10, rationalizing the numerator or denominator *did not change* the value of the original fraction since we were merely multiplying by the number 1. Rationalizing the numerator of a fraction just "exchanges" a radical in the top for a radical in the bottom. Likewise, rationalizing the denominator "exchanges" a radical in the bottom for a radical in the top.

Exercises

Reduce to lowest terms:

1. $\dfrac{x^2 + x - 2}{x^2 - 4}$

2. $\dfrac{x^3 + 2x^2 - 3x}{-2x^2 - x + 3}$

3. $\dfrac{2a^2 - 3ab - 9b^2}{2ab^2 + 3b^3}$

4. $\dfrac{x^4 - 4x^2}{x^3 + 2x^2}$

Perform the indicated operation(s) and reduce to lowest terms:

5. $\dfrac{2a^2 - 2b^2}{b - a} \cdot \dfrac{4a + 4b}{a^2 + 2ab + b^2}$

6. $\dfrac{xy^2}{wz^2} \cdot \dfrac{zw}{x^2 z}$

7. $\dfrac{x^2 - 6x + 9}{x^2 - x - 6} \cdot \dfrac{3x + 6}{2x^2 - 7x + 3}$

8. $5x^3 y \cdot \dfrac{16x^2 - 9y^2}{8x^2 + 6xy}$

9. $\dfrac{\left(\dfrac{y^2 - 2y - 15}{y^2 - 9} \right)}{\left(\dfrac{12 - 4y}{y^2 - 6y + 9} \right)}$

10. $\dfrac{r^2 st}{\left(\dfrac{r + sr}{9r^3 t} \right)}$

11. $\dfrac{\left(\dfrac{25xy^6}{14x + 2y} \right)}{2xy}$

12. $\dfrac{\left(\dfrac{3x^2 + 2x - 1}{2x + 6} \right)}{\left(\dfrac{x^2 - 1}{x^2 + 2x - 3} \right)}$

13. $\dfrac{4y + 3}{y - 2} - \dfrac{y - 2}{y - 2}$

14. $\dfrac{3}{x} - \dfrac{8}{-x}$

15. $\dfrac{3x}{4y} - \dfrac{4y}{3x}$

16. $\dfrac{2x - 1}{4 - x} + \dfrac{x + 2}{3x - 12}$

17. $\dfrac{x + 5}{x^2 + 7x + 10} - \dfrac{x - 1}{x^2 + 5x + 6}$

18. $\dfrac{1}{x - 2} + \dfrac{1}{x - 2} - \dfrac{x}{x^2 - 4}$

19. $\dfrac{4yz}{x^2} - \dfrac{2z}{xy^2} + \dfrac{1}{xyz}$

20. $\dfrac{1 + \dfrac{1}{x}}{1 - \dfrac{1}{x}}$

21. $\dfrac{\dfrac{x^2 - y^2}{xy}}{\dfrac{x - y}{y}}$

22. $\dfrac{\dfrac{1}{a} + \dfrac{1}{b}}{\dfrac{a^2 - b^2}{ab}}$

23. $\dfrac{\dfrac{x}{1 - x} + \dfrac{1 + x}{x}}{\dfrac{1 - x}{x} + \dfrac{x}{1 + x}}$

24. $\dfrac{x^{-3} - x}{x^{-2} - 1}$

25. $\dfrac{2x(x + 1)^{-\frac{1}{2}} - (x + 1)^{\frac{1}{2}}}{x^2}$

26. $\dfrac{(2x + 1)^{\frac{1}{2}} - (x + 2)(2x + 1)^{-\frac{1}{2}}}{2x + 1}$

Rationalize the numerator of each expression:

27. $\dfrac{1 + \sqrt{x + 2}}{\sqrt{x + 2}}$

28. $\sqrt{9x^2 + x} - 3x$

29. $\dfrac{\sqrt{y}}{\sqrt{y} - 1}$

30. $\dfrac{3 - \sqrt{x}}{5}$

Rationalize the denominator of each expression:

31. $\dfrac{1 + \sqrt{x + 2}}{\sqrt{x + 2}}$

32. $\dfrac{\sqrt{y}}{\sqrt{y} - 1}$

33. $\dfrac{1}{\sqrt{x} + \sqrt{y}}$

34. $\dfrac{10}{3\sqrt{x}}$

Inequalities

Statements containing $<$, $>$, $\leq$ or $\geq$ are called **inequalities**. A **solution** of an inequality is a number that makes the statement true when the variable is replaced by that number.

Example 1: The solution set for the inequality $x \geq -1$ is every real number greater than or equal to negative one. Below is a graph of this inequality. The interval notation for this solution set is $[-1, \infty)$.
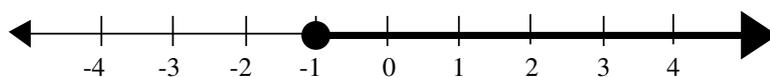
Note that we show -1 is a solution to the inequality by using a closed circle at -1. If -1 was not a solution (i.e. if the inequality was $x > -1$), we would use an open circle and the interval notation for the solution set would be $(-1, \infty)$.

Given any two real numbers, a and b, exactly one of the following is true:

$$a < b,\ a = b,\ \text{or}\ a > b.$$

To solve inequalities like

$$-7x - 8 < 6 \qquad \text{and} \qquad 9 + 3(2x - 4) \geq 2x + 13$$

we use methods similar to those used to find solutions of equations. These methods are based on the four rules below.

Rules of Inequalities:

Let a, b, and c be any real numbers.

(1) If $a < b$ and $b < c$, then $a < c$.

(2) If $a < b$, then $a + c < b + c$.

(3) If $a < b$ and c is positive, then $ac < bc$.

(4) If $a < b$ and c is negative, then $ac > bc$.

Example 2:

(a) $2 < 3$ and $3 < 4$ so $2 < 4$

(b) $5 < 7$ so $5 + 2 < 7 + 2 \Rightarrow 7 < 9$

(c) $3 < 8$ so $(3)(4) < (8)(4) \Rightarrow 12 < 32$

(d) $7 < 9$ so $(7)(-2) > (9)(-2) \Rightarrow -14 > -18$

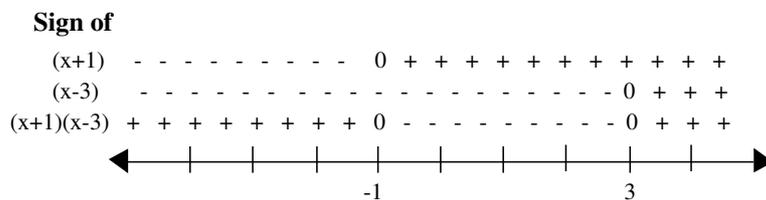**Example 3:** Solve (a) $-7x - 8 < 6$ and (b) $9 + 3(2x - 4) \geq 2x + 13$ for x and write your solution set in interval notation.

$$
\begin{array}{ll}
\text{(a)} \quad -7x - 8 < 6 & \text{(b)} \quad 9 + 3(2x - 4) \geq 2x + 13 \\[4pt]
\quad -7x - 8 + 8 < 6 + 8 & \quad 6x - 3 \geq 2x + 13 \\[4pt]
\quad -7x < 14 & \quad 6x - 2x - 3 + 3 \geq 2x - 2x + 13 + 3 \\[4pt]
\quad \dfrac{-7x}{-7} > \dfrac{14}{-7} & \quad 4x \geq 16 \\[4pt]
\quad x > -2 & \quad \dfrac{4x}{4} \geq \dfrac{16}{4} \\[4pt]
\quad (-2, \infty) & \quad x \geq 4 \\[4pt]
& \quad [4, \infty)
\end{array}
$$

A **polynomial inequality** is an inequality that can be written with a polynomial on one side and 0 on the other. After we have written the polynomial inequality with 0 on one side, we can begin to solve it by factoring the polynomial into linear factors. One way to find the solution set for a polynomial inequality is to develop a chart containing the signs of these factors.

Example 4: Find the solution set for $x^2 - 3 > 2x$ and write it in interval notation.

$$
x^2 - 3 > 2x \quad \Rightarrow \quad x^2 - 2x - 3 > 0 \quad \Rightarrow \quad (x - 3)(x + 1) > 0
$$

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Since we are interested in values that make the product $(x-3)(x+1)$ positive, the solution set is $(-\infty, -1) \cup (3, \infty)$.

Example 5: Find the solution set for $20x - x^2 - x^3 \leq 0$ and write it in interval notation.

$$20x - x^2 - x^3 \leq 0 \quad \Rightarrow \quad x(20 - x - x^2) \leq 0 \quad \Rightarrow \quad x(4-x)(x+5) \leq 0$$

Sign of

```
x              - - - - - - - - - - - - - - - - 0 + + + + + + + + + + + +
(4-x)          + + + + + + + + + + + + + + + + + + + + + + + + 0 - - -
(x+5)          - - - - 0 + + + + + + + + + + + + + + + + + + + + + + +
x(4-x)(x+5)    + + +  0 - - - - - - - - - - -  0 + + + + + + +  0 - - -
```


Since we are interested in values that make the product $x(4-x)(x+5)$ zero or negative, the solution set is $[-5, 0] \cup [4, \infty)$.

A **rational inequality** is an inequality that can be written with a rational function (a polynomial over a polynomial) on one side and 0 on the other. We can solve rational inequalities using a method similar to the one used for polynomial inequalities.

Example 6: Find the solution set for $\dfrac{x^2 + 14x + 49}{x - 2} \geq 0$ and write it in interval notation.

$$\frac{x^2 + 14x + 49}{x - 2} \geq 0 \quad \Rightarrow \quad \frac{(x+7)^2}{x-2} \geq 0$$

Sign of

```
(x+7)          -  - 0 + + + + + + + + + + + + + + + + + + + + + + + + +
(x+7)          -  - 0 + + + + + + + + + + + + + + + + + + + + + + + + +
(x-2)          - - - - - - - - - - - - - - - - - - - - - - 0 + + + + +
(x+7)²/(x-2)   - - 0 - - - - - - - - - - - - - - - - - - - U + + + + +
```


The solution set is $\{-7\} \cup (2, \infty)$. Note the "U" in the sign chart. This signifies that our expression is undefined at 2. *BEWARE!* In general, a rational expression is not defined for values which make the denominator zero.

To solve inequalities involving absolute value, we need the following properties:

> **Properties of Inequalities with Absolute Values:**
>
> Let a be any real number and let $f(x)$ be an expression involving x.
>
> (1) If $|f(x)| < a$, then $-a < f(x) < a$.
>
> (2) If $|f(x)| > a$, then $f(x) < -a$ or $f(x) > a$.

Note that if $|f(x)| < a$, then $f(x)$ satisfies both inequalities $f(x) > -a$ *and* $f(x) < a$ for all values of x, and thus we may write this as one statement: $-a < f(x) < a$. However, if $|f(x)| > a$, then, given a specific x value, $f(x)$ satisfies *either* $f(x) < -a$ *or* $f(x) > a$ but *not* both. Thus we *cannot* write this as a single statement without absolute value signs.

Example 7: Solve $|4 + x| - 6 > 5$ and express the solution set in interval notation.

$$|4 + x| - 6 > 5$$
$$|4 + x| - 6 + 6 > 5 + 6$$
$$|4 + x| > 11$$
$$4 + x < -11 \quad or \quad 4 + x > 11$$
$$x < -15 \quad or \quad x > 7$$

The solution set is $(-\infty, -15) \cup (7, \infty)$.

Example 8: Solve $|7 - 2x| \leq 9$ and express the solution set in interval notation.

$$|7 - 2x| \leq 9$$
$$-9 \leq 7 - 2x \leq 9$$
$$-16 \leq -2x \leq 2$$
$$8 \geq x \geq -1$$

The solution set is $[-1, 8]$.

Exercises

Find the values of x that satisfy the following inequalities and write the solution sets in interval notation.

1. $\frac{1}{2}x + 3 > 2$

2. $-3x + 24 \geq 0$

3. $2x - 7 < 5x - 9$

4. $-\frac{5}{6}x \leq -\frac{3}{4}$

5. $3x - \frac{1}{8} > \frac{3}{8} + 2x$

6. $\frac{13x + 7}{4} \leq 5x - 9$

7. $x^2 + 4x > 0$

8. $x^4 - x^5 > 0$

9. $3x^2 + 2x - 8 \leq 0$

10. $9x^2 < 81$

11. $12 - 5x - 2x^2 \leq 0$

12. $x^3 + 4x > 5x^2$

13. $\frac{x + 4}{x} < 0$

14. $\frac{3 - x}{x + 2} \leq 0$

15. $\frac{x^2 - 7x}{x + 1} \geq 0$

16. $\frac{1}{x} < \frac{1}{6}$

17. $\frac{x^2 - 4}{6} - \frac{2x - 3}{4} > \frac{1}{12}$

18. $\frac{x}{x - 5} \leq \frac{1}{11}$

19. $|2x - 3| > 7$

20. $|4 - x| \geq 8$

21. $|5x + 13| \leq 12$

22. $\left| \frac{5}{9} + \frac{1}{3}x \right| < \frac{1}{6}$

23. $2|2x - 7| + 11 \geq 25$

24. $4|5 - 3x| < 32$

Functions and Lines

Functions and Function Notation

One of the most important concepts in mathematics is that of a **function**, which can be defined as a rule that assigns to each element x of a set A a *unique* value y in a set B. We will write $y = f(x)$ as function notation. A function can be thought of as a device that takes an input x and produces an output y in such a way that every time a given value of x is put into the device, the same value of $y = f(x)$ is always produced.

Example 1: $f(x) = x^2 + 4$ is a function.

Note, however, that $x = 2$ and $x = -2$ produce the same output $y = 8$. So a function may give the same *output* for two different *inputs*, but it may not give two different *outputs* for the same *input*.

In general, any polynomial expression can be used to define a function:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.$$

Example 2: Let f take people in your class as input and produce as output the given person's biological mother. So if Bob is in your class, $f(\text{Bob})$ would be Bob's mother. Then (provided cloning is not a possibility!) f is a function.

Example 3: Suppose g is a rule that takes as input the people in your class and produces as output a parent of a given person. Then, since $g(\text{Bob})$ might be Bob's mother one time and his father the next time, g is not a function.

Example 4: Suppose h is a rule that takes as input the natural numbers (positive integers) and produces as output for a given x a real number whose square is x. Then $h(9)$ could be 3 or -3, and h can't be a function.

Domain and Range

The **domain** of a function $y = f(x)$ is the set of values which can be used as *inputs* for the function, i.e., the x's. The **range** of a function is the set of possible *outputs* for the function, i.e., the y's. In first-year calculus, we are interested in real functions of a real variable, so the domain and range will be sets of real numbers. If the domain is not

specified for a function, it is assumed to be the largest set of real numbers for which the function is defined.

Example 5: A polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

is defined for all real values of x, so the domain is the set of all real numbers.

Example 6: The range of a polynomial function may not be quite so easy to determine. The range of $g(x) = x^3 - 1$, for example, is all real numbers (WHY?), but the range for $h(x) = x^2 + 3$ is the interval $[3, \infty)$.

Example 7: A function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, is called a **rational function**. The domain is the largest set of real numbers for which the fraction is defined. This is the set of values for which the denominator $q(x)$ is not 0. For example, the domain of the rational function $f(x) = \frac{x^2+1}{x-5}$ must be

$$\{x | x \neq 5\} = (-\infty, 5) \cup (5, \infty).$$

Example 8: We saw in Example 7 that *division by 0 is not allowed.* Similarly, *we cannot take even roots of negative numbers* and produce a real number. So the domain of the function $g(x) = \sqrt{x - 6}$ is $[6, \infty)$.

Example 9: To find the domain of the function $h(x) = \frac{\sqrt{x+7}}{x^2-16}$, we need both

$$(1) x + 7 \geq 0 \qquad \text{and} \qquad (2) x^2 - 16 \neq 0.$$

So $x \geq -7$ and $x \neq \pm 4$, and the domain is $[-7, -4) \cup (-4, 4) \cup (4, \infty)$.

Equality of Functions

Two functions $f(x)$ and $g(x)$ are **equal** if *both* of the following are true:

    1) f and g have the same domain

    2) $f(x) = g(x)$ for all x in the domain of f and g.

As the following example shows, we must check both rules before declaring that two functions are equal.

Example 10: Are $f(x) = \frac{(x+1)(x-2)}{x-2}$ and $g(x) = x + 1$ equal?

If $x \neq 2$, then $f(x) = \frac{(x+1)(x-2)}{x-2} = x + 1 = g(x)$.

But if $x = 2$, then $g(2) = 3$ and $f(2) = \frac{3 \cdot 0}{0}$, which does not exist.

So f and g are not equal, since they do not have the same domain.

Piecewise Functions

A **piecewise function** is just what it sounds like: a function that is defined differently on different parts of its domain. A familiar example of a piecewise function is the **absolute value function**, which is defined by

$$f(x) = |x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

So $|2| = 2$, since $2 \geq 0$, but $|-1000| = -(-1000) = 1000$, because $-1000 < 0$.

Example 11: If f is the piecewise function defined by

$$f(x) = \begin{cases} 2x^2 + 1 & \text{if} \quad x > 1 \\ 0 & \text{if} \quad x = 1 \\ 5x - 4 & \text{if} \quad x < 1 \end{cases}$$

then $f(4) = 2(4)^2 + 1 = 32 + 1 = 33$, since $4 > 1$. Likewise, $f(1) = 0$, and $f(0) = 5(0) - 4 = -4$.

Example 12: Let $g(x)$ be a function that takes as input people in your class, and uses the following rule for output:

$$g(x) = \begin{cases} x\text{'s first name if } x \text{ is male} \\ x\text{'s last name if } x \text{ is female} \end{cases}$$

Then $g(\text{Bob Smith}) = \text{Bob}$ and $g(\text{Sally Jones}) = \text{Jones}$.

The Cartesian Plane and Graphing Functions

In most of the examples above, we have described a function by using words or equations. Another very useful way to define a function is to display it as a graph. The

graph of a function $y = f(x)$ is the set of all points (x, y) in the Cartesian plane where x is in the domain of f and $y = f(x)$.

The definition of a function requires that for every x in the domain, there is one and *only one* value y with $y = f(x)$. Now a vertical line in the plane goes through all the points that have a specific x-value. This means that if a curve in the plane represents a function, a vertical line can intersect the curve in at most one point because a given x-value can have *only one* y-value associated with it. This is so useful in identifying graphs of functions that we state it as a rule:

Vertical Line Test: A curve in the Cartesian plane represents a function if and only if any vertical line passes through the curve at most one time.

Example 13: The graph on the left represents a function, while the graph on the right fails the Vertical Line Test and cannot be a function.
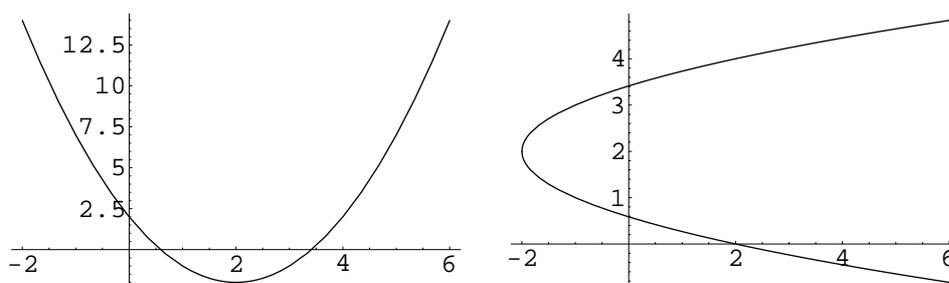
Warning about Functions

It is tempting to try to simplify expressions involving functions by using some of the familiar properties of arithmetic. This can lead to *lots of mistakes and wrong answers*. Here is a list of things to remember for functions in general:

$$f(a + b) \neq f(a) + f(b)$$

$$f(a - b) \neq f(a) - f(b)$$

$$f(ab) \neq f(a)f(b)$$

$$f\left(\frac{a}{b}\right) \neq \frac{f(a)}{f(b)}$$

$$f(ab) \neq af(b)$$

$$f\left(\frac{a}{b}\right) \neq \frac{f(a)}{b}$$

Example 14: If $f(x) = \sqrt{x}$, then

$$f(25 - 16) = f(9) = \sqrt{9} = 3,$$

but

$$f(25) - f(16) = \sqrt{25} - \sqrt{16} = 5 - 4 = 1.$$

Remember that to simplify a function, you must *first use the definition of the function*, and then use the properties that are familiar from arithmetic, algebra, or trigonometry.

Example 15: If $f(x) = x^3 + 5$, find $f(a + b)$.

$$f(a + b) = (a + b)^3 + 5 = a^3 + 3a^2b + 3ab^2 + b^3 + 5$$

Equations of Lines

The simplest types of lines are horizontal or vertical. A **horizontal line** is the set of of all points with the same y-value: $y = c$. Similarly, a **vertical line** is the set of all points with the same x-values: $x = k$. In general, we know from algebra that any line that is not vertical can be represented by an equation of the form $y = mx + b$, where m and b are real numbers. Since $mx + b$ is a polynomial, a non-vertical line is just a particular type of function.

Geometrically, we think of lines in terms of points and slope. If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in the Cartesian plane with $x_1 \neq x_2$, the **slope** of the line joining P_1 and P_2 is the number $m = \frac{y_2 - y_1}{x_2 - x_1}$. A line may be thought of as a curve with the property that the slope between any two points on the curve is a fixed constant. We can use this property to write equations for lines in the plane that are determined by different types of information.

1. If we are given a slope m and a point $P_1(x_1, y_1)$ on the line, the equation of the line through P_1 with slope m is determined by the fact that if $P(x, y)$ is another point on

the line, the slope between any two points must be m. So the equation of the line is $m = \frac{y - y_1}{x - x_1}$, which simplifies to

$$y - y_1 = m(x - x_1).$$

We call this the **point-slope form** of an equation of a line.

2. The **y-intercept** of a line is the number b such that the line crosses the y-axis at $y = b$. This means that the point $(0, b)$ is on the line, and the equation for a line with y-intercept b and slope m would be $y - b = m(x - 0)$, or

$$y = mx + b.$$

We call this the **slope-intercept form** of an equation of a line.

3. A line can also be determined by two points. If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points on a line, the slope of the line must be $m = \frac{y_2 - y_1}{x_2 - x_1}$. So the equation of the line is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, or

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

We call this the **two-point form** of an equation of a line.

Example 16: The line through the point $(3,2)$ that has slope 4 has equation $y - 2 = 4(x - 3)$, which simplifies to $y = 4x - 10$.

Example 17: The line through the two points $(1,4)$ and $(3,5)$ has equation $\frac{y-4}{x-1} = \frac{5-4}{3-1}$, which gives $y - 4 = \frac{1}{2}(x - 1)$, or $y = \frac{1}{2}x + \frac{7}{2}$.

Example 18: The line $y = 4x - 7$ has slope $m = 4$ and y-intercept $b = -7$.

Example 19: What is the x-intercept of the line $y = 3x - 7$?

The x-intercept of a line is the point where the line crosses the x-axis, and on the x-axis, we must have $y = 0$. If $y = 0$ in the equation of the line, then $0 = 3x - 7$, and $x = \frac{7}{3}$ is the x-intercept for the line.

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Graphs of Lines

The simplest approach to drawing the graph of a line is to find two points on the line and draw a straight line through them. A point on the line can be determined by choosing a value for x and using the equation of the line to find the corresponding value for y.

Example 20: Sketch the graph of the line $y = \frac{3}{4}x - 6$.

If $x = 0$, then $y = -6$. If $x = 4$, then $y = -3$. So the points $(0, -6)$ and $(4, -3)$ are on the line, and the graph is shown below:
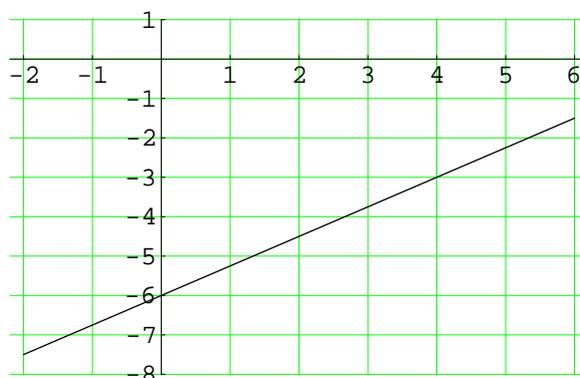
Parallel and Perpendicular Lines

Two lines are **parallel** if they never cross. *Parallel lines must have the same slope.* So the lines $y = m_1 x + b_1$ and $y = m_2 x + b_2$ are parallel if $m_1 = m_2$.

Two lines are **perpendicular** if they cross at right angles. *The slopes of perpendicular lines must be negative reciprocals.* In other words, the lines $y = m_1 x + b_1$ and $y = m_2 x + b_2$ are perpendicular if $m_2 = -\frac{1}{m_1}$.

Example 21: What is the line that is parallel to the line $y = 3x + 5$ and passes through the point $(2,4)$?

The slope of the line $y = 3x + 5$ is $m = 3$, so a line parallel to it has the same slope. The line through $(2,4)$ with slope $m = 3$ is $y - 4 = 3(x - 2)$, giving us an answer of $y = 3x - 2$.

Example 22: Find an equation for the line with y-intercept 4 that is perpendicular to the line $y = 2x + 3$.

The slope of the given line is $m_1 = 2$, so the slope of the desired line will be $m_2 = -\frac{1}{m_1} = -\frac{1}{2}$. The equation of the line with slope $-\frac{1}{2}$ and y-intercept 4 is $y = -\frac{1}{2}x + 4$.

Exercises

Which of the following are functions? Explain why or why not in each case. If the given rule defines a function, give the domain of the function.

1. $f(x) = y$ such that $y^4 = x$

2. $f(x) = 5x^4 - 7x^3$

3. $f(x) = y$ such that $y^3 = x$

4. The rule that takes each year as input and gives back the age in years you were during that calendar year.

5. The rule that takes an age in years as input and gives back the calendar year in which you were that age.

6. The rule that takes people in your class as input and gives back each person's age in years at noon on September 4, 2001.

7. $f(x)$ is the largest prime number less than x

Find the domain of each of the following functions.

8. $f(x) = \sqrt{x + 12}$

9. $f(x) = \frac{x+2}{x^2-4}$

10. $f(x) = \frac{\sqrt{x^2+4}}{x^3-27}$

11. $f(x) = \frac{x+4}{\sqrt{x-1}(x^2-5x+6)}$

Which of the following graphs represent functions?

12.
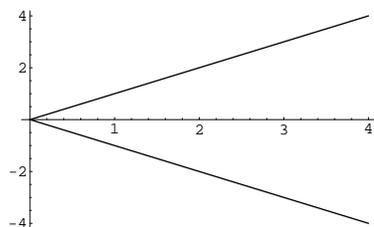
13.
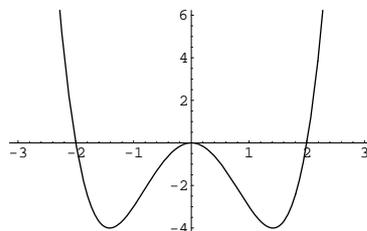
14.
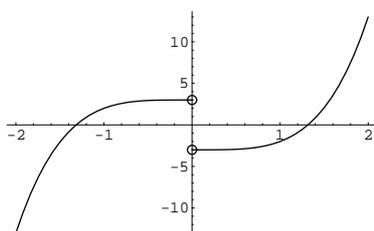
15.
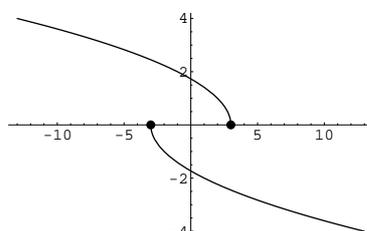
For each of the following piecewise functions, give the value of $f(x)$ for each of the x's listed.

16. $x = -3,\ 1,\ 2$

$$f(x) = \begin{cases} \frac{1}{x^2-1} & \text{if } x \neq 1, -1 \\ 1 & \text{if } x = 1, -1 \end{cases}$$

17. $x = -1,\ 0,\ 3$

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \leq -1 \\ 3x & \text{if } -1 < x < 1 \\ 2x - 1 & \text{if } x \geq 1 \end{cases}$$

18. $x = $ and, blue, ingot

$$f(x) = \begin{cases} \text{the first letter} & \text{if the word } x \text{ starts with a vowel} \\ \text{the last letter} & \text{if the word } x \text{ starts with a consonant} \end{cases}$$

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19. $x = -2, 0, 1, 3$

$$f(x) = \begin{cases} x^2 + 3 & \text{if } x > 1 \\ \frac{1}{x+2} & \text{if } -2 < x < 1 \\ x & \text{if } x \le -2 \end{cases}$$

Find an equation for each of the following lines. In each case, solve the equation for y.

20. The line with slope 2 and y-intercept 3.

21. The line with slope $-\frac{5}{6}$ and y-intercept 3.

22. The line that goes through the points (2,1) and (3,2).

23. The line that goes through the points (4,2) and (6,1).

24. The line through the point (1,0) that is parallel to the line $y = \frac{x}{3} + 2$.

25. The line that is parallel to $y = -x - 1$ and goes through the point (0,4).

26. The line that is perpendicular to $y = 2x$ and has y-intercept 5.

27. The line that is perpendicular to $y = -3x + 2$ and goes through the point (2,2).

28. The line that is perpendicular to $y = x + 2$ and intersects that line at the same point that it intersects with $y = 2x + 1$.

Graph each of the following lines.

29. $y = -3x + 2$

30. $y = -\frac{1}{2}x + 1$

31. $y = \sqrt{2}x - 3$

32. $y - 4 = \frac{\sqrt{3}}{2}x$

Completing the Square

A **quadratic expression** in x is a second-degree polynomial which has the general form

$$ax^2 + bx + c.$$

In certain calculus applications, we need to be able to write quadratic expressions as the sum or difference of two squares. The process for doing this, called **completing the square**, involves reversing the procedure for squaring out a binomial:

$$(x + d)^2 = x^2 + 2dx + d^2.$$

Here is an example that shows the steps to follow when $a = 1$.

Example 1: Complete the square on the quadratic $x^2 + 6x - 5$.

1. Group the terms containing x and x^2.

$$x^2 + 6x - 5 = (x^2 + 6x) - 5$$

2. Take the coefficient of x, divide it by 2 and square the result.

$$\left(\frac{6}{2}\right)^2 = 3^2 = 9$$

3. Add the number from step 2 inside the parentheses and subtract it outside.

$$x^2 + 6x - 5 = (x^2 + 6x + 9) - 5 - 9$$

4. Write the expression inside the parentheses as a square and simplify the numerical term outside.

$$x^2 + 6x - 5 = (x + 3)^2 - 14$$

Example 2: Complete the square in the expression $y^2 - 7y + 21$.

$$y^2 - 7y + 21 = (y^2 - 7y) + 21$$
$$= \left(y^2 - 7y + \frac{49}{4}\right) + 21 - \frac{49}{4} \qquad \text{since } \left(\frac{-7}{2}\right)^2 = \frac{49}{4}$$
$$= \left(y - \frac{7}{2}\right)^2 + \frac{84 - 49}{4}$$
$$= \left(y - \frac{7}{2}\right)^2 + \frac{35}{4}$$

Example 3: Complete the square on x and y in the equation $x^2 + 12x + y^2 - y = 14$.

$$14 = x^2 + 12x + y^2 - y$$
$$= (x^2 + 12x) + (y^2 - y)$$
$$= (x^2 + 12x + 36) - 36 + \left(y^2 - y + \frac{1}{4}\right) - \frac{1}{4}$$
$$= (x + 6)^2 + \left(y - \frac{1}{2}\right)^2 - \left(36 + \frac{1}{4}\right)$$
$$= (x + 6)^2 + \left(y - \frac{1}{2}\right)^2 - \frac{145}{4}$$

So we have

$$(x + 6)^2 + \left(y - \frac{1}{2}\right)^2 - \frac{145}{4} = 14,$$

which simplifies to the equation

$$(x + 6)^2 + \left(y - \frac{1}{2}\right)^2 = 14 + \frac{145}{4}$$

or

$$(x + 6)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{56 + 145}{4} = \frac{201}{4}.$$

A quadratic expression where $a \neq 1$ requires an additional step in completing the square.

Example 4: Complete the square in the expression $4x^2 - 48x + 5$.

1. Group the terms containing x^2 and x.

$$4x^2 - 48x + 5 = \left(4x^2 - 48x\right) + 5$$

2. Factor out the coefficient a of x^2 from the terms in the parentheses.

$$4x^2 - 48x + 5 = 4\left(x^2 - 12x\right) + 5$$

3. Take the coefficient of x, divide it by 2, and square the result.

$$\left(\frac{-12}{2}\right)^2 = 36$$

4. Add the number from Step 3 inside the parentheses and *subtract a times the number outside the parentheses.*

$$4x^2 - 48x + 5 = 4\left(x^2 - 12x + 36\right) + 5 - 4(36)$$

5. Write the expression inside the parentheses as a square and simplify the numerical expression outside the parentheses.

$$4x^2 - 48x + 5 = 4(x - 6)^2 - 139$$

Example 5: Complete the square for $1 + 15m - 10m^2$.

$$\begin{aligned}
1 + 15m - 10m^2 &= -10m^2 + 15m + 1 \\
&= \left(-10m^2 + 15m\right) + 1 \\
&= -10\left(m^2 - \frac{3}{2}m\right) + 1 \\
&= -10\left(m^2 - \frac{3}{2}m + \frac{9}{16}\right) + 1 - (-10)\left(\frac{9}{16}\right) \\
&= -10\left(m - \frac{3}{4}\right)^2 + \frac{53}{8}
\end{aligned}$$

Example 6: Complete the square for the expression $14x - x^2$.

$$\begin{aligned}
14x - x^2 &= (-x^2 + 14x) \\
&= (-1)(x^2 - 14x) \\
&= (-1)(x^2 - 14x + 49) - (-1)(49) \\
&= 49 - (x - 7)^2
\end{aligned}$$

Exercises

Complete the square in each of the following expressions:

1. $x^2 + 10x - 20$
2. $6y^2 - 24y$
3. $\frac{191}{16} - \frac{1}{2}w - w^2$
4. $4x^2 + 2x - 12$
5. $9 - 5k^2 + 22k$
6. $-6n^2 - 18n - 13$

Complete the square in x and y for each of the following equations:

7. $9x^2 - 18x + 4y^2 + 4y = 27$
8. $2y^2 - 6x + 4y + 7 - x^2 = 0$
9. $16x^2 - 9y^2 = 305 + 90y - 64x$
10. $12x - 2y^2 - 3x^2 - 4y + 8 = 0$
11. $y - x = 1 - 2x^2 - 2y^2$

Trigonometry

The Unit Circle

The first key to understanding trigonometry is to know the unit circle. The **unit circle** is the circle centered at $(0,0)$ with radius 1.
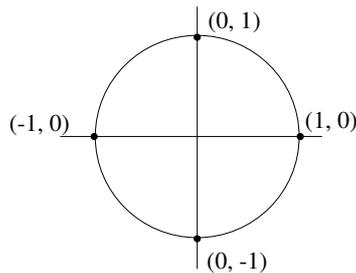
Consider an angle θ in the unit circle. The angle is positive if it is measured counterclockwise from the positive x-axis and negative if it is measured clockwise.
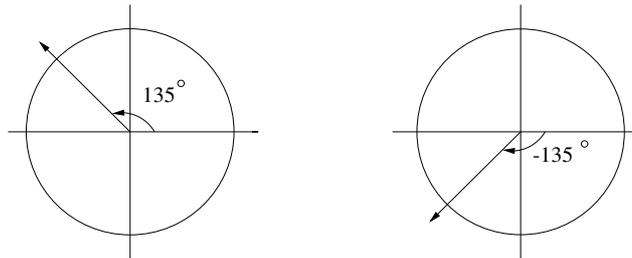
The above angles are measured using **degrees**. An angle θ may also be measured using **radians**. The radian measurement corresponds to a distance around the circumference, C, of the unit circle ($C = 2\pi$).

Let us measure an arc on the unit circle starting at $(1,0)$ of length $\frac{\pi}{4}$ and ending at a point P. If we draw a ray from the origin through point P, we have formed an angle θ, where $\theta = \frac{\pi}{4}$ radians.
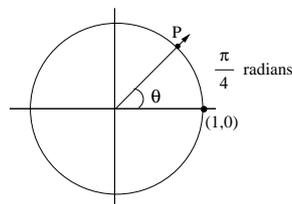
The following two angles are the same.
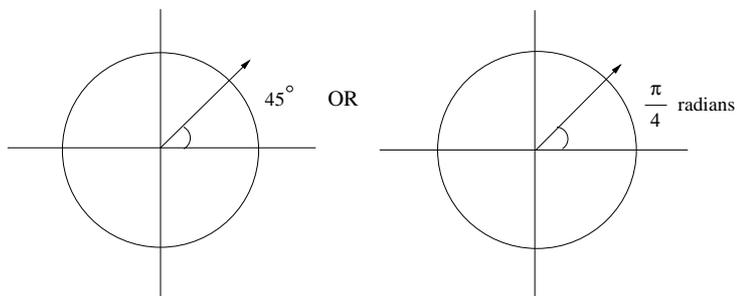
To convert radians to degrees and vice versa, use the following equation.

$$\boxed{\pi \text{ radians} = 180°}$$

Example 1: Convert 30° to radians.

$$\pi \text{ radians} = 180°$$
$$\frac{\pi}{180} \text{ radians} = 1°$$
$$30 \cdot \frac{\pi}{180} \text{ radians} = 30 \cdot 1°$$
$$\frac{\pi}{6} \text{ radians} = 30°$$

Example 2: Convert $\frac{8\pi}{5}$ radians to degrees.

$$\pi \text{ radians} = 180°$$
$$\frac{8}{5} \cdot \pi \text{ radians} = \frac{8}{5} \cdot 180°$$
$$\frac{8\pi}{5} \text{ radians} = 288°$$

Trigonometric Functions

Consider the point $P = (x, y)$ where the angle of measure θ intersects the unit circle. We use the coordinates x and y of this point to define six **trigonometric functions** of θ.
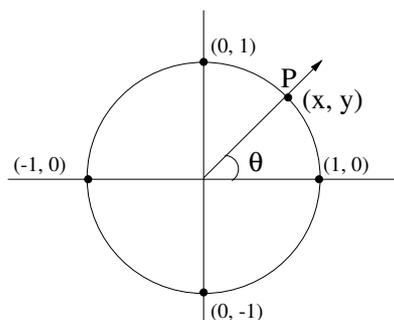
We define the **cosine** of θ to be the x-coordinate of this point and the **sine** of θ to be the y-coordinate of this point. We use the abbreviations "cos" for cosine and "sin" for sine. Thus,

$$\cos \theta = x \qquad \text{and} \qquad \sin \theta = y.$$

Notice that the x and y coordinates of all points on the unit circle lie somewhere between -1 and 1. Thus

$$-1 \le \cos \theta \le 1 \quad \text{and} \quad -1 \le \sin \theta \le 1$$

no matter what θ is.

We can use geometry to determine $\sin \frac{\pi}{4}$ and $\cos \frac{\pi}{4}$. Since $\frac{\pi}{4}$ radians equals $45°$, consider a $45°$-$45°$-$90°$ triangle with a hypotenuse of length 1. Such a triangle must have legs each of length $\frac{1}{\sqrt{2}}$.

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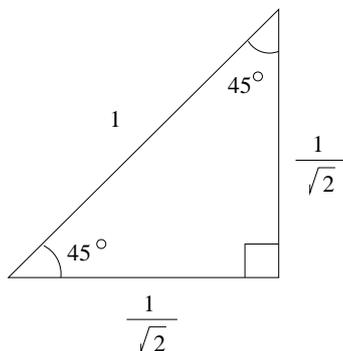
By moving this triangle into the unit circle and remembering that $45°$ equals $\frac{\pi}{4}$ radians, we find that the corresponding point on the unit circle has (x, y)-coordinates $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.
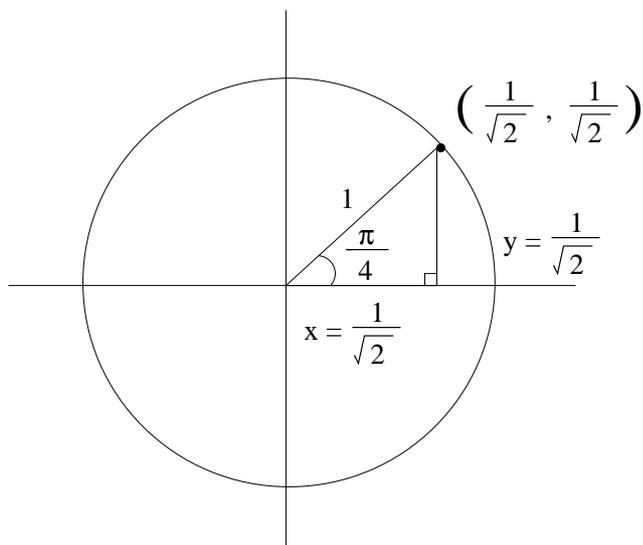
Thus

$$\sin\frac{\pi}{4} = y = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \cos\frac{\pi}{4} = x = \frac{1}{\sqrt{2}}.$$

We can find $\sin\frac{\pi}{3}$, $\cos\frac{\pi}{3}$, $\sin\frac{\pi}{6}$, and $\cos\frac{\pi}{6}$ in a similar fashion by noting that $\frac{\pi}{3}$ radians equals $60°$ and $\frac{\pi}{6}$ radians equals $30°$. A $30°$-$60°$-$90°$ triangle with a hypotenuse of length 1 gives us the information we need.

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$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2} \right)$$

$$\frac{\pi}{6} \quad 1 \quad y = \frac{1}{2}$$

$$x = \frac{\sqrt{3}}{2}$$

$$\left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$$

$$1 \quad \frac{\pi}{3} \quad y = \frac{\sqrt{3}}{2}$$

$$x = \frac{1}{2}$$
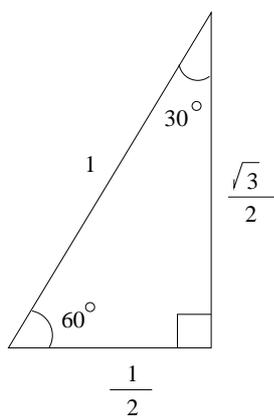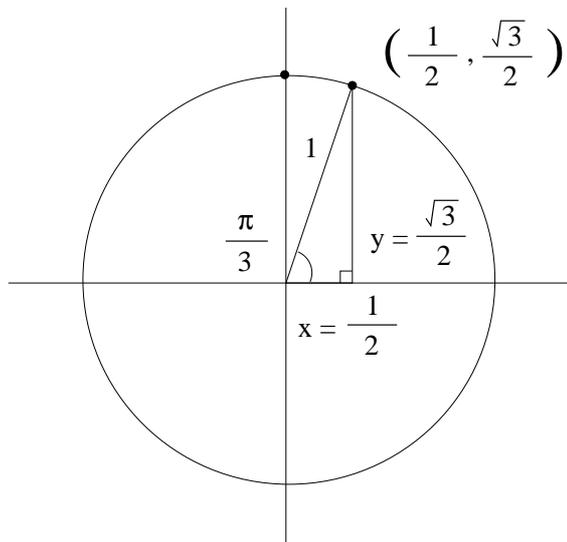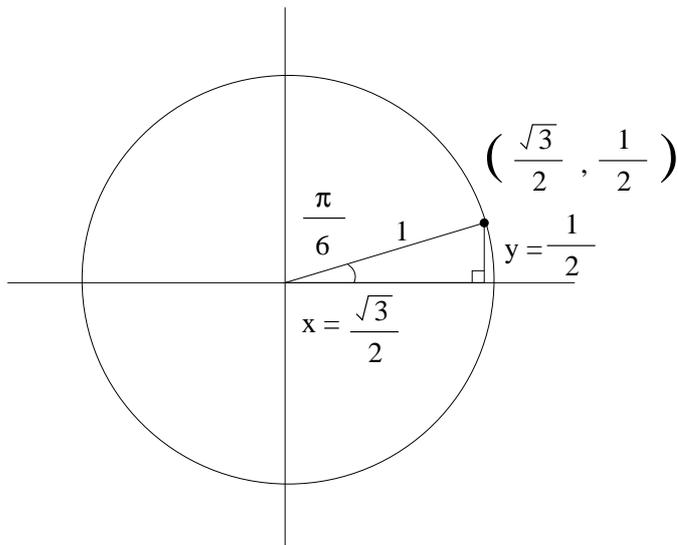
Thus

$$\sin \frac{\pi}{6} = \frac{1}{2} \qquad \text{and} \qquad \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

and

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \text{and} \qquad \cos \frac{\pi}{3} = \frac{1}{2}.$$

In calculus, we measure angles in *radians*, and we often use the trig values we just found, so it will be helpful to memorize the "enhanced" unit circle on the next page.
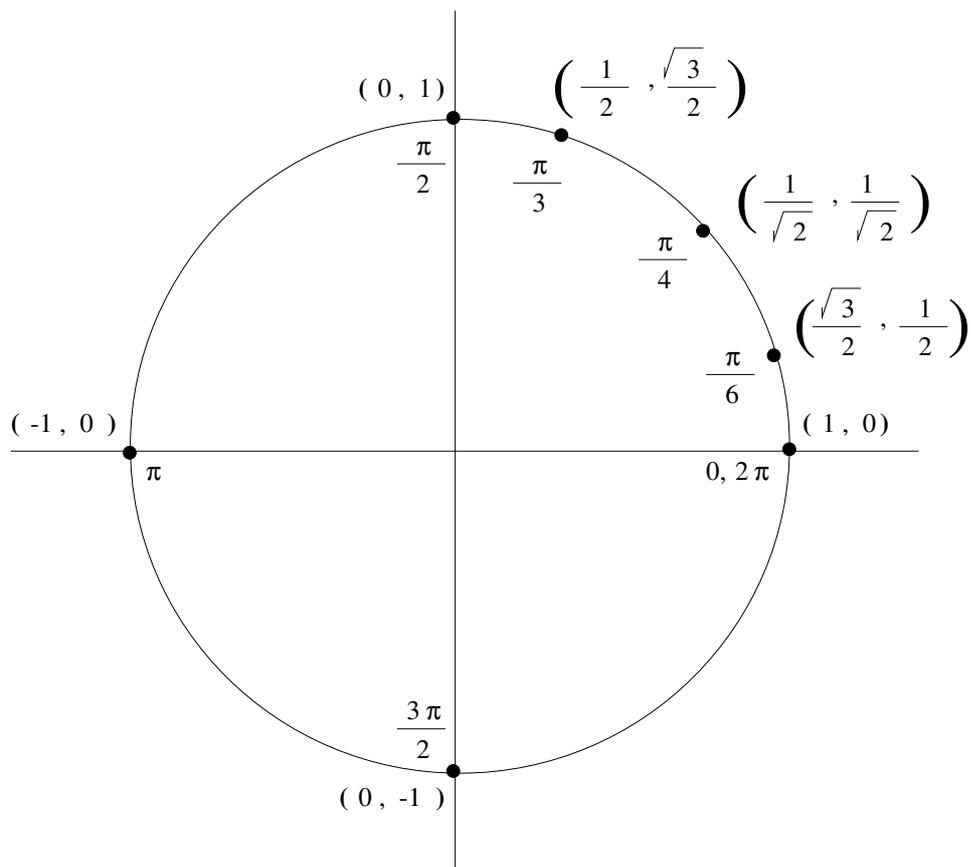
Four other trigonometric functions are defined using sine and cosine. They are the **secant** ("sec"), **cosecant** ("csc"), **tangent** ("tan"), and **cotangent** ("cot") trigonometric functions, defined as follows.

$$\sec\theta = \frac{1}{\cos\theta} \qquad\qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad\qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Since $\cos\theta$ and $\sin\theta$ are 0 for some values of θ, the trig functions $\sec\theta$, $\csc\theta$, $\tan\theta$, and $\cot\theta$ are undefined for some values of θ. For more information, see the graphs of the trig functions at the end of this section.

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Example 3: Find $\cot \frac{\pi}{6}$.

$$\cot \frac{\pi}{6} = \frac{\cos \frac{\pi}{6}}{\sin \frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

The unit circle has four quadrants.
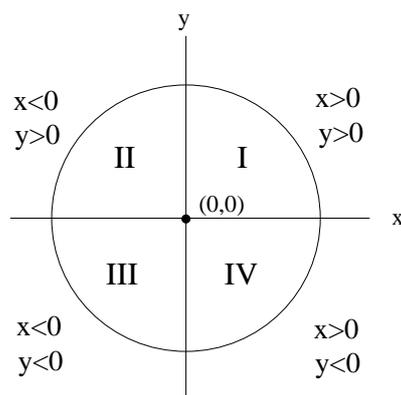
Note that a point in Quad II will have a negative x-value and a positive y-value. Thus, cosine is negative and sine is positive in Quad II. To remember which trig functions are positive in which quadrants, use the saying, "<u>A</u>ll <u>S</u>tudents <u>T</u>ake <u>C</u>alculus."
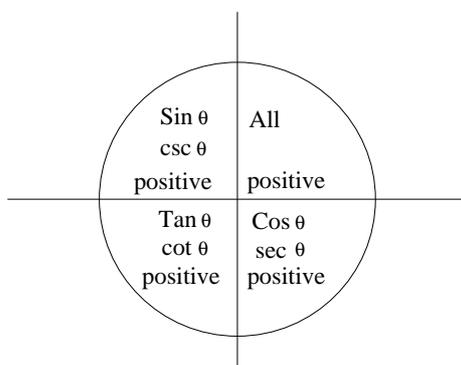
Example 4: Find $\cos \frac{4\pi}{3}$.

We will first find in which quadrant $\frac{4\pi}{3}$ lies.
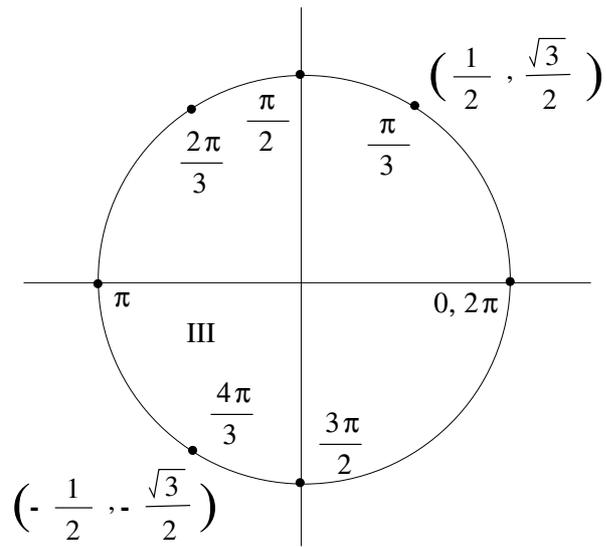
Due to symmetry, $\frac{\pi}{3}$ and $\frac{4\pi}{3}$ have the same coordinates except for the negative signs. Thus

$$\cos \frac{4\pi}{3} = -\frac{1}{2}.$$

Trigonometric Identities

The following trigonometric identities will be useful in calculus.

$$\cos^2 \theta + \sin^2 \theta = 1$$
$$\tan^2 \theta + 1 = \sec^2 \theta$$
$$\cot^2 \theta + 1 = \csc^2 \theta$$

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$
$$\sin(a - b) = \sin a \cos b - \cos a \sin b$$
$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$
$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$
$$= 2 \cos^2 \theta - 1$$
$$= 1 - 2 \sin^2 \theta$$

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$
$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

$$\cos(-\theta) = \cos \theta$$
$$\sin(-\theta) = -\sin \theta$$

It is *not* necessary to memorize all of the above identities! By knowing how an identity is derived, one can reduce the amount of memorization necessary.

The equation of the unit circle is $x^2 + y^2 = 1$. The Pythagorean theorem also gives us the equation $x^2 + y^2 = 1$. This equation gives us

$$\cos^2 \theta + \sin^2 \theta = 1.$$

Dividing this equation by $\cos^2 \theta$, we obtain

$$\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \qquad \text{or} \qquad 1 + \tan^2 \theta = \sec^2 \theta.$$

Dividing $\cos^2 \theta + \sin^2 \theta = 1$ by $\sin^2 \theta$ we obtain

$$\frac{\cos^2 \theta}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} \qquad \text{or} \qquad \cot^2 \theta + 1 = \csc^2 \theta.$$

Using

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

and

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

with $a = b = \theta$, we obtain the formulas

$$\sin 2\theta = 2 \sin \theta \cos \theta \qquad \text{and} \qquad \cos 2\theta = \cos^2 \theta - \sin^2 \theta.$$

Also

$$\begin{aligned}
\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\
&= \cos^2 \theta - (1 - \cos^2 \theta) \\
&= 2 \cos^2 \theta - 1
\end{aligned}$$

and

$$\begin{aligned}
\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\
&= (1 - \sin^2 \theta) - \sin^2 \theta \\
&= 1 - 2 \sin^2 \theta
\end{aligned}$$

using the identity $\cos^2 \theta + \sin^2 \theta = 1$. We can then solve for $\cos^2 \theta$ in $\cos 2\theta = 2 \cos^2 \theta - 1$ to get

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}.$$

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Solving for $\sin^2\theta$ in $\cos 2\theta = 1 - 2\sin^2\theta$ gives us

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}.$$

Finding the Values of Trigonometric Functions

Given the values for one trig function, we can find the values for the other five trig functions. There are two ways to do this.

Method 1: Use identities.

Example 5: (Using Method 1) Given θ is in Quad III and $\cot\theta = 2$, find the values of the remaining trig functions.

First, cotangent and tangent are reciprocals, so

$$\tan\theta = \frac{1}{\cot\theta} = \frac{1}{2}.$$

Next use the identity $\cot^2\theta + 1 = \csc^2\theta$ to get $\csc\theta$:

$$\cot^2\theta + 1 = \csc^2\theta$$
$$2^2 + 1 = \csc^2\theta$$
$$\csc\theta = \pm\sqrt{5}$$
$$\csc\theta = -\sqrt{5}$$

since θ is in Quad III.

Cosecant and sine are reciprocals, so

$$\sin\theta = \frac{1}{\csc\theta} = -\frac{1}{\sqrt{5}}.$$

Using $\sin^2\theta + \cos^2\theta = 1$, we get

$$\cos^2\theta + \left(-\frac{1}{\sqrt{5}}\right)^2 = 1$$

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$$\cos^2\theta = \frac{4}{5}$$

$$\cos\theta = \pm\frac{2}{\sqrt{5}}$$

$$\cos\theta = -\frac{2}{\sqrt{5}}$$

since θ is in Quad III.

Cosine and secant are reciprocals, so

$$\sec\theta = \frac{1}{\cos\theta} = -\frac{\sqrt{5}}{2}.$$

This gives us all six trig function values.

Method 2: Use a right triangle with 0 radians $< \theta < \frac{\pi}{2}$ radians.
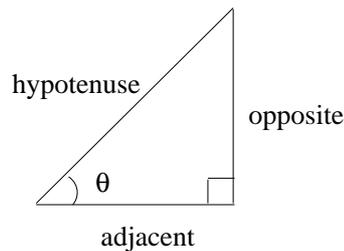
This gives

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

One can remember these equations using the acronym "SOH CAH TOA" where S = sine, O = opposite, H = hypotenuse, C = cosine, A = adjacent, and T = tangent.

(In the previous triangle, we assumed θ was in Quad I. However, by using symmetry, we can assume θ is in any quadrant. But be careful of the SIGNS of the trig functions when θ is in Quad II, III, or IV!)

We will now work Example 5 again using Method 2.

Example 5: (Using Method 2) Given θ is in Quad III and $\cot\theta = 2$, find the values of the remaining trig functions.

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adj}}{\text{opp}} = \frac{2}{1}$$

We then have the following triangle.

$$1^2 + 2^2 = h^2$$
$$h = \sqrt{5}$$
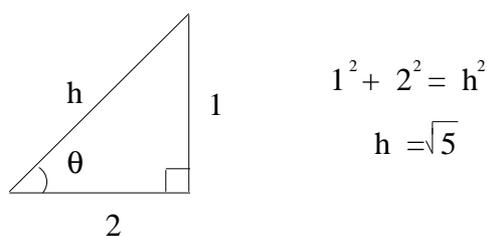
Next note that θ is in Quad III. Thus

$$\cos\theta = -\frac{\text{adj}}{\text{hyp}} = -\frac{2}{\sqrt{5}} \qquad \sec\theta = \frac{1}{\cos\theta} = -\frac{\sqrt{5}}{2}$$

$$\sin\theta = -\frac{\text{opp}}{\text{hyp}} = -\frac{1}{\sqrt{5}} \qquad \csc\theta = \frac{1}{\sin\theta} = -\sqrt{5}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{2}$$

This gives us the values of all six trig functions.

Solving Equations Involving Trig Functions

We will begin by considering equations with one term involving a trigonometric function.

Example 6: Solve $2\sin x = 1$.

We want to find which values of x make this equation true. We will *not* rewrite this equation as $x = \cdots$. We will isolate the trig function instead:

$$\sin x = \frac{1}{2}$$

To find the solutions to this equation, we find the radian value(s) that will give us sine equal to $\frac{1}{2}$.

For x in $[0, 2\pi]$, we have 2 solutions: $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

However, if x can be any number, note that $\frac{\pi}{6} + 2\pi$, $\frac{\pi}{6} + 4\pi$, $\frac{\pi}{6} - 2\pi$, and $\frac{5\pi}{6} + 2\pi$ are also solutions. In fact, for x in $(-\infty, \infty)$, we have an infinite number of solutions that can be represented as

$$x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + 2n\pi$$

where n is any integer.

Example 7: Solve $2 \sin 2x = -1$ for x in $[0, 2\pi]$.

We will begin by solving for $\sin 2x$.

$$\sin 2x = -\frac{1}{2}$$

We want solutions x for which

$$0 \le x \le 2\pi \quad \text{or} \quad 0 \le 2x \le 4\pi.$$

For which radian values θ between 0 and 4π does $\sin \theta$ equal $-\frac{1}{2}$?

Between 0 and 2π, $\sin \theta = -\frac{1}{2}$ for

$$\theta = \frac{7\pi}{6} \quad \text{and} \quad \theta = \frac{11\pi}{6}.$$

Between 2π and 4π, $\sin \theta = -\frac{1}{2}$ for

$$\theta = \frac{7\pi}{6} + 2\pi = \frac{19\pi}{6} \quad \text{and} \quad \theta = \frac{11\pi}{6} + 2\pi = \frac{23\pi}{6}.$$

To compensate for the $2x$ in $\sin 2x$, we will set $\theta = 2x$.

$$\sin \frac{7\pi}{6} = -\frac{1}{2} = \sin 2x \qquad \Rightarrow \qquad 2x = \frac{7\pi}{6} \qquad \Rightarrow \qquad x = \frac{7\pi}{12}$$

Similarly,

$$\sin \frac{11\pi}{6} = -\frac{1}{2} = \sin 2x \qquad \Rightarrow \qquad x = \frac{11\pi}{12},$$

$$\sin \frac{19\pi}{6} = -\frac{1}{2} = \sin 2x \qquad \Rightarrow \qquad x = \frac{19\pi}{12},$$

and

$$\sin \frac{23\pi}{6} = -\frac{1}{2} = \sin 2x \qquad \Rightarrow \qquad x = \frac{23\pi}{12}.$$

Thus the solutions for x in $[0, 2\pi]$ are

$$x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}.$$

We will now consider equations with more than one term involving a trigonometric function. The key concept involved in these types of equations is factoring.

Example 8: Solve $2\cos^2 x \tan x - \tan x = 0$ for x in $[0, 2\pi]$.

We begin by factoring $\tan x$ out of each term.

$$\tan x(2\cos^2 x - 1) = 0.$$

Thus either

$$\tan x = 0 \qquad \text{or} \qquad 2\cos^2 x - 1 = 0.$$

If $\tan x = 0$, then
$$x = 0, \pi, 2\pi.$$

If $2\cos^2 x - 1 = 0$, then
$$\cos^2 x = \tfrac{1}{2}$$
and $\cos x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$.
So for x in $[0, 2\pi]$,
$$x = \tfrac{\pi}{4}, \tfrac{3\pi}{4}, \tfrac{5\pi}{4}, \tfrac{7\pi}{4}.$$

The solution set is

$$\left\{ 0, \pi, 2\pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\}.$$

In some problems we will first use identities and then factor.

Example 9: Solve $\cos x = \cos 2x$ for x in $[0, 2\pi]$.

We will first use the identity $\cos 2x = 2\cos^2 x - 1$.

$$\cos x = 2\cos^2 x - 1$$

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$$2\cos^2 x - 1 - \cos x = 0$$

Next, factor.

$$2\cos^2 x - \cos x - 1 = 0$$

$$(2\cos x + 1)(\cos x - 1) = 0$$

Thus either

$$2\cos x + 1 = 0 \qquad \text{or} \qquad \cos x - 1 = 0.$$

If $2\cos x + 1 = 0$, then
$$\cos x = -\tfrac{1}{2}$$
and $x = \frac{2\pi}{3}, \frac{4\pi}{3}$.

If $\cos x - 1 = 0$, then
$$\cos x = 1$$
and $x = 0, 2\pi$.

So the solution set is

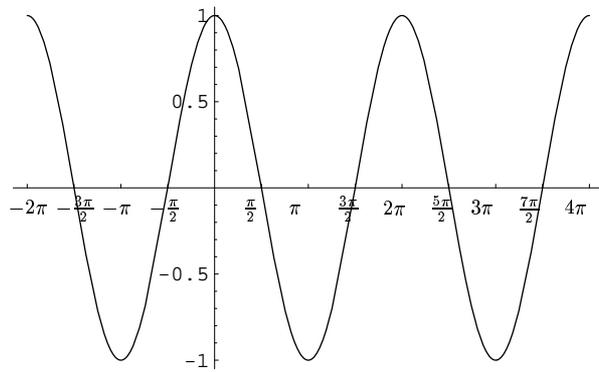$$\left\{ 0, 2\pi, \frac{2\pi}{3}, \frac{4\pi}{3} \right\}.$$
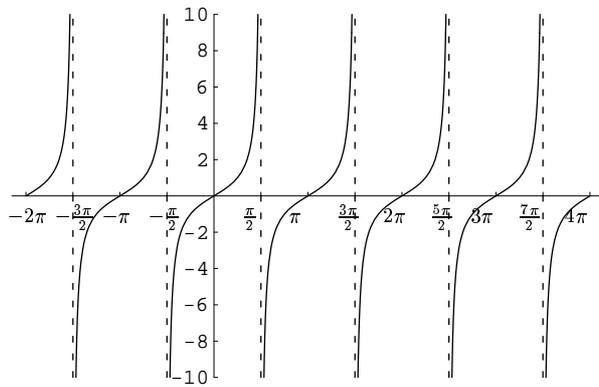
Graphs of Trig Functions

The graphs of the six trig functions are shown below. The trig functions are all periodic. (A function is **periodic** with period p if $f(x + p) = f(x)$ for all real numbers x. Such functions repeat every p units along the x-axis.) Sine and cosine have periods of 2π. Tangent, cotangent, secant, and cosecant all have periods of π.
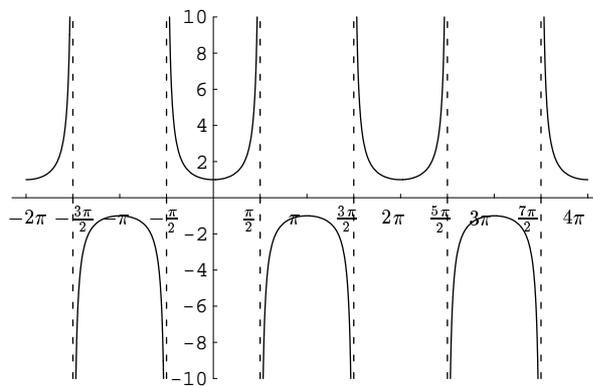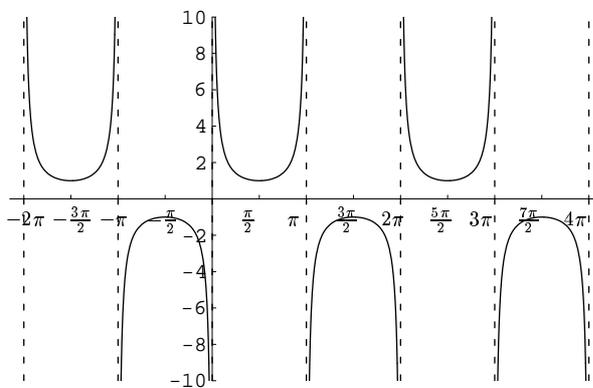

$$y = \sin x$$

$y = \cos x$

$y = \tan x$

$y = \cot x$

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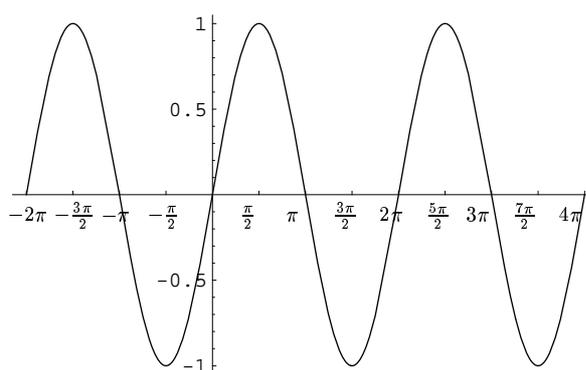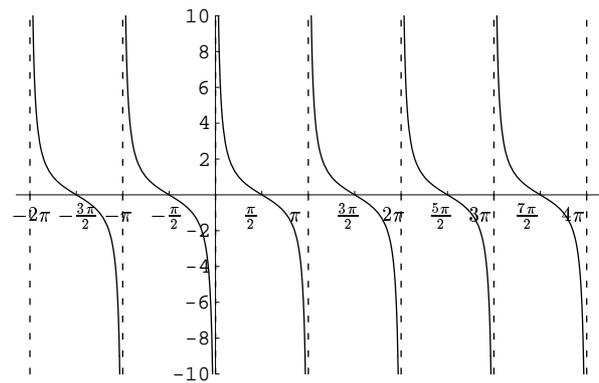
$$y = \sec x$$

$$y = \csc x$$

Exercises

Change to radian measure:

1. $50°$
2. $120°$
3. $375°$
4. $-12°$

Change to degree measure:

5. $-\frac{5\pi}{6}$
6. $\frac{35\pi}{12}$
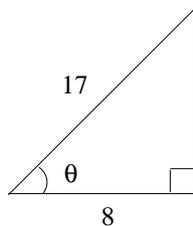7. $\frac{7\pi}{8}$
8. $-\frac{2\pi}{3}$

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Find the function values:

9. $\sin \frac{5\pi}{3}$ 　　　10. $\tan \frac{\pi}{6}$ 　　　11. $\csc \frac{11\pi}{4}$ 　　　12. $\cos \left(-\frac{2\pi}{3}\right)$

13. $\sec \frac{11\pi}{6}$ 　　　14. $\sin \left(-\frac{3\pi}{2}\right)$ 　　　15. $\cot \frac{5\pi}{4}$ 　　　16. $\cos \frac{5\pi}{6}$

A function value and a quadrant are specified. Find the other five function values.

17. $\sin \theta = \frac{1}{3}$, II 　　　　18. $\sec \theta = \frac{5}{3}$, I

19. $\tan \theta = 5$, III 　　　　20. $\cot \theta = -4$, IV

21. Find the six trigonometric function values for the following θ:

Solve, finding all solutions:

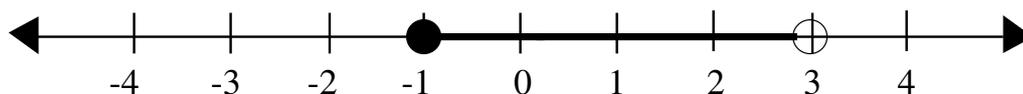22. $\tan x = \sqrt{3}$ 　　　　23. $2\cos^2 x = 1$ 　　　　24. $2\sin^2 x - 5\sin x + 2 = 0$

Solve, finding all solutions in $[0, 2\pi]$.

25. $\sec^2 x - 4 = 0$ 　　　26. $2\sin^3 x = \sin x$ 　　　27. $\cos 2x \sin x + \sin x = 0$

28. $\sec^2 x = 4\tan^2 x$ 　　　29. $\cos 2x - \sin x = 1$

Answers to Odd-Numbered Exercises

1   Numbers and Absolute Values

1. set notation: $\left\{ x \,\middle|\, -1 \le x < 3 \right\}$; number line:

3. interval notation: $(-4, -1] \cup (0, 3)$; set notation: $\left\{ x \,\middle|\, -4 < x \le -1 \text{ or } 0 < x < 3 \right\}$

5. $x = 1, 5$

7. $x = -2, -1$

2   Exponents and Radicals

1. $x^{13/3}$

3. 32

5. $16x^4$

7. 4

9. 2

11. 49

13. 1

15. 1024

17. $\left(\dfrac{x}{2}\right)^{yz}$

19. $\dfrac{y^3}{8}$

21. $x^{24}y^{20}$

23. $-4x^{7/6}yz^{-5/3}$

25. $\dfrac{64b^{28}c^8}{a^{16}}$

27. $3\pi - \sqrt{17x}$; $9\pi^2 - 17x$

29. $-2 - x\sqrt{3+y}$; $4 - x^2(3+y) = 4 - 3x^2 - x^2y$

3   Polynomials and Factoring

1. 2

3. 0

5. $2x^4 - 2x$

7. $5x^2 + 29x - 6$

9. $x^2 + 4x + 4$

11. $-6x^4 - 2x^3 + 5x^2 + x - 1$

13. $acx^2 + (ad + bc)x + bd$

15. $x^3 + 3x^2 + 2x$

17. $x(x+1)$

19. $2x(x+1)^2$

21. $(4x - 5)(4x + 5)$ 23. $(2 - x)(4 + 2x + x^2)$

25. $(x + 3)(x + 9)$ 27. $(x - 4)(x + 2)$

29. $(x - 1)(x + 1)^2$ 31. $(x + 2)(x - 1)(x^2 + x + 1)$

33. $x = -11,\ 1$ 35. $x = \pm 3$

37. $x = \frac{1 \pm \sqrt{5}}{2}$ 39. $x = 2 \pm \sqrt{5}$

41. no real roots 43. $x = 6$

45. $x = \frac{1}{2},\ -2,\ 3$ 47. $x = -4$

49. $x = \pm 2,\ \pm \sqrt{3}$ 51. $x = -3,\ -1,\ 0,\ 4$

4 Algebraic Fractions

1. $\dfrac{x - 1}{x - 2}$ 3. $\dfrac{a - 3b}{b^2}$ 5. -8

7. $\dfrac{3}{2x - 1}$ 9. $\dfrac{y - 5}{-4}$ 11. $\dfrac{25y^5}{28x + 4y}$

13. $\dfrac{3y + 5}{y - 2}$ 15. $\dfrac{9x^2 - 16y^2}{12xy}$ 17. $\dfrac{4}{(x + 2)(x + 3)}$

19. $\dfrac{4y^3z^2 - 2xz^2 + xy}{x^2y^2z}$ 21. $\dfrac{x + y}{x}$ 23. $\dfrac{1 + x}{1 - x}$

25. $\dfrac{x - 1}{x^2(x + 1)^{1/2}}$ 27. $\dfrac{-1 - x}{\sqrt{x + 2} - (x + 2)}$ 29. $\dfrac{y}{y - \sqrt{y}}$

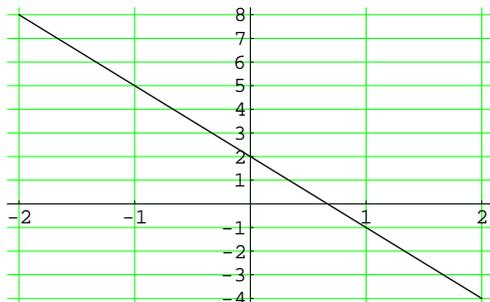31. $\dfrac{\sqrt{x + 2} + x + 2}{x + 2}$ 33. $\dfrac{\sqrt{x} - \sqrt{y}}{x - y}$

5 Inequalities

1. $(-2, \infty)$
3. $\left(\frac{2}{3}, \infty\right)$
5. $\left(\frac{1}{2}, \infty\right)$
7. $(-\infty, -4) \cup (0, \infty)$
9. $\left[-2, \frac{4}{3}\right]$
11. $(-\infty, -4] \cup \left[\frac{3}{2}, \infty\right)$
13. $(-4, 0)$
15. $(-1, 0] \cup [7, \infty)$
17. $(-\infty, 0) \cup (3, \infty)$
19. $(-\infty, -2) \cup (5, \infty)$
21. $\left[-5, -\frac{1}{5}\right]$
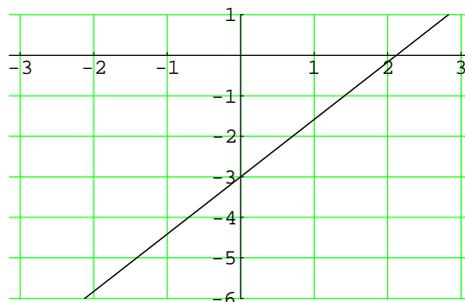23. $(-\infty, 0] \cup [7, \infty)$

6 Functions and Lines

1. not a function because $f(1)$ could be -1 or 1

3. function: $\sqrt[3]{x}$ is the only number that when cubed equals x

   domain $= (-\infty, \infty)$

5. not a function: there are *two* calendar years in which you were any given age

7. function: there can be no more than one *largest* prime less than a given number

   domain $= [2, \infty)$

9. $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

11. $(1, 2) \cup (2, 3) \cup (3, \infty)$

13. function

15. not a function

17. $f(-1) = -1$, $f(0) = 0$, $f(3) = 5$

19. $f(-2) = -2$, $f(0) = \frac{1}{2}$, $f(1)$ is undefined, $f(3) = 12$

21. $y = -\frac{5}{6}x + 3$

23. $y = -\frac{1}{2}(x - 4) + 2$

25. $y = -x + 4$

27. $y = \frac{1}{3}(x - 2) + 2$

29.   31.

7   Completing the Square

1. $(x + 5)^2 - 45$

3. $12 - (w + \frac{1}{4})^2$

5. $\frac{166}{5} - 5(k - \frac{11}{5})^2$

7. $9(x - 1)^2 + 4(y + \frac{1}{2})^2 = 37$

9. $16(x + 2)^2 - 9(y + 5)^2 = 144$

11. $2(x - \frac{1}{4})^2 + 2(y + \frac{1}{4})^2 = \frac{5}{4}$

8   Trigonometry

1. $\frac{5\pi}{18}$ 3. $\frac{25\pi}{12}$

5. $-150°$ 7. $157.5°$

9. $-\frac{\sqrt{3}}{2}$ 11. $\sqrt{2}$

13. $\frac{2}{\sqrt{3}}$ 15. 1

17. $\cos\theta = -\frac{2\sqrt{2}}{3}$, $\tan\theta = -\frac{1}{2\sqrt{2}}$, $\sec\theta = -\frac{3}{2\sqrt{2}}$, $\cot\theta = -2\sqrt{2}$, $\csc\theta = 3$

19. $\cos\theta = -\frac{1}{\sqrt{26}}$, $\sin\theta = -\frac{5}{\sqrt{26}}$, $\sec\theta = -\sqrt{26}$, $\csc\theta = -\frac{\sqrt{26}}{5}$, $\cot\theta = \frac{1}{5}$

21. $\cos\theta = \frac{8}{17}$, $\sin\theta = \frac{15}{17}$, $\tan\theta = \frac{15}{8}$, $\sec\theta = \frac{17}{8}$, $\csc\theta = \frac{17}{15}$, $\cot\theta = \frac{8}{15}$

23. $\left\{\frac{\pi}{4} + 2n\pi, \frac{3\pi}{4} + 2n\pi, \frac{5\pi}{4} + 2n\pi, \frac{7\pi}{4} + 2n\pi\right\}$ or $\left\{\frac{\pi}{4} + \frac{n}{2}\pi\right\}$

25. $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$ 27. $\left\{0, \pi, 2\pi, \frac{\pi}{2}, \frac{3\pi}{2}\right\}$

29. $\left\{0, \pi, 2\pi, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$