Analysis of topological charge in electrodynamic systems using Fourier decomposition

John Wikswo[+] and Mark-Anthony Bray[*]
Living State Physics Group, Vanderbilt University
[+]*Department of Physics and Astronomy*
[*]*Department of Biomedical Engineering*

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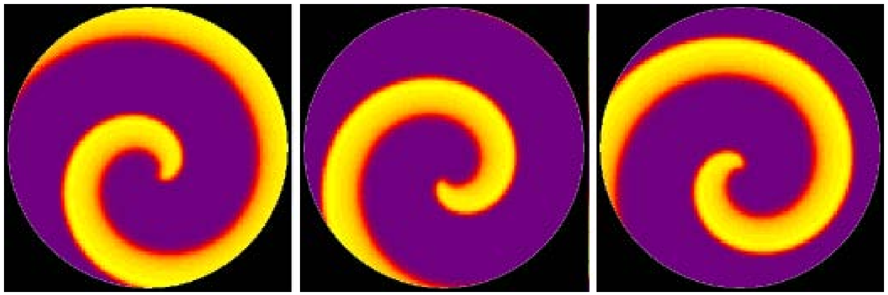
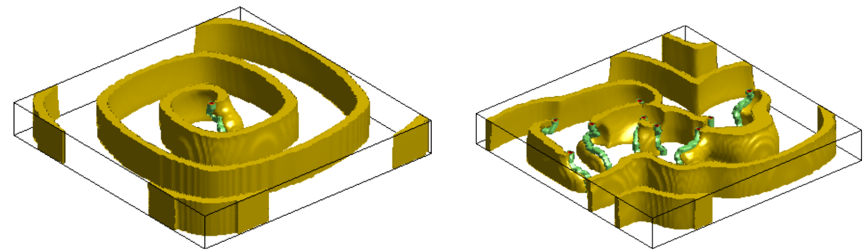
Spiral and Scroll Waves in Nature

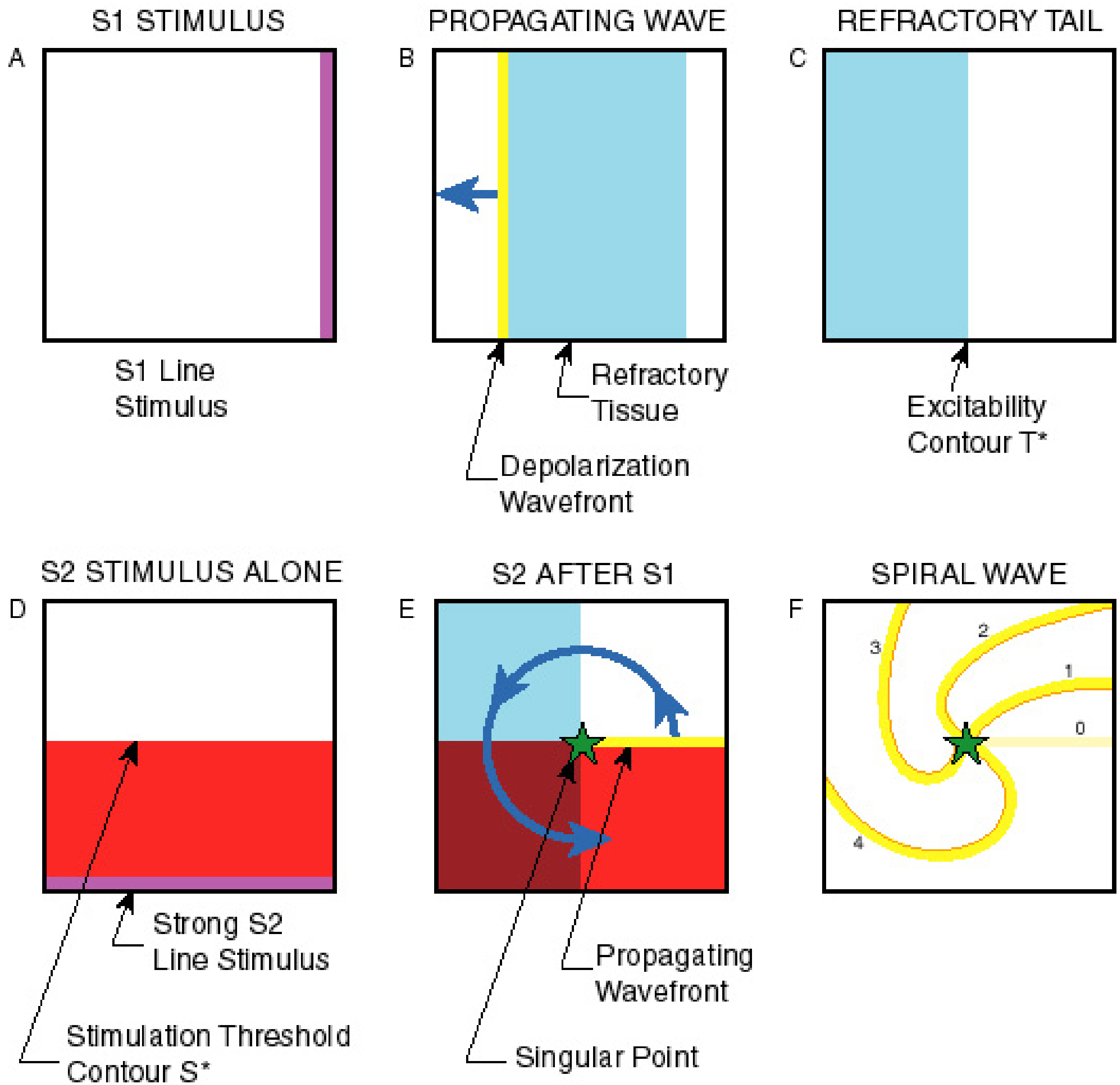




- A generic property of excitable media
- Have been shown to occur in
  - Circulating waves of bioelectric activity in cardiac and retinal tissue
  - Autocatalytic chemical reactions, such as Belousov-Zhabotinsky reaction (BZ)
  - cAMP waves in slime mold *Dictyostelium discoideum*
  - Intracellular calcium release in oocytes
  - Oxidation of CO on crystal surfaces in ultrahigh vacuum conditions
- Cardiac fibrillation is multiple scroll waves in 3-D

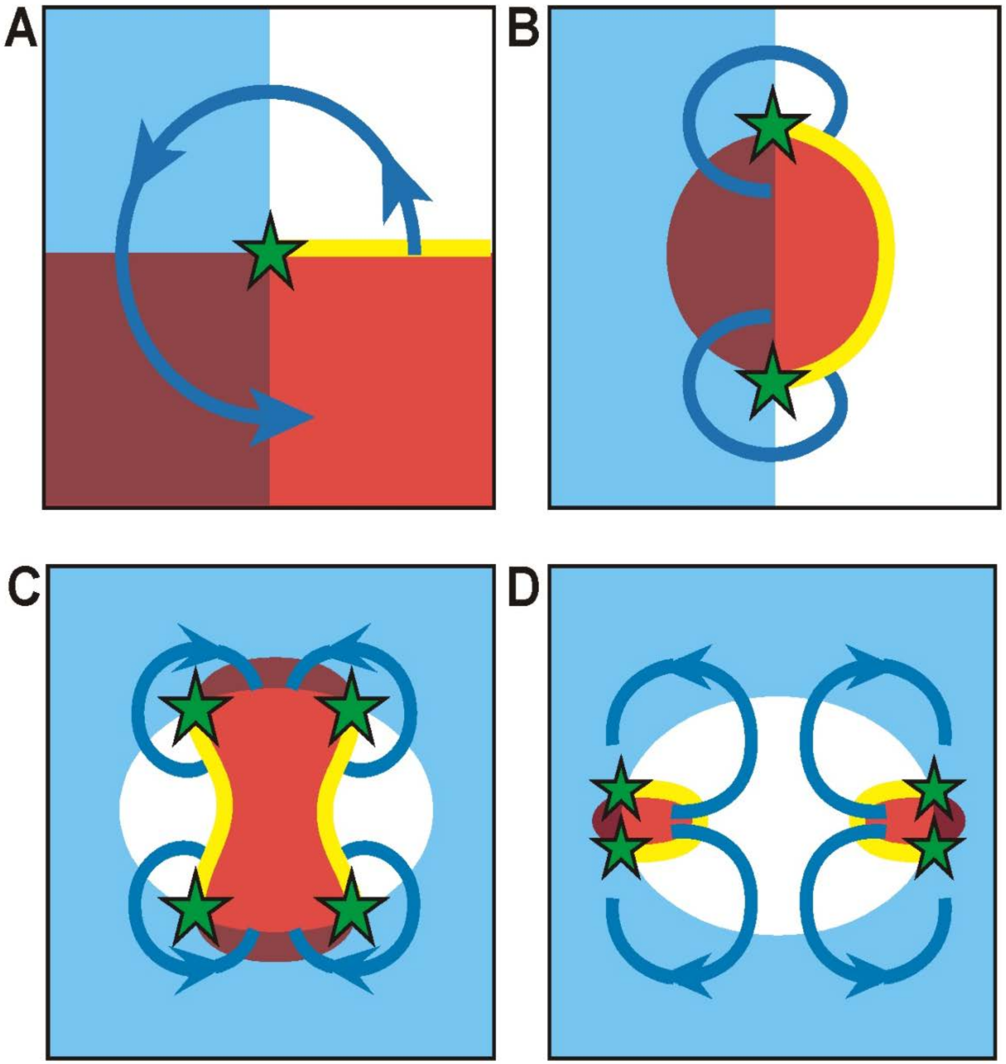
Initiation of Spiral Wave Reentry

S1-S2 crossed-field stimulation



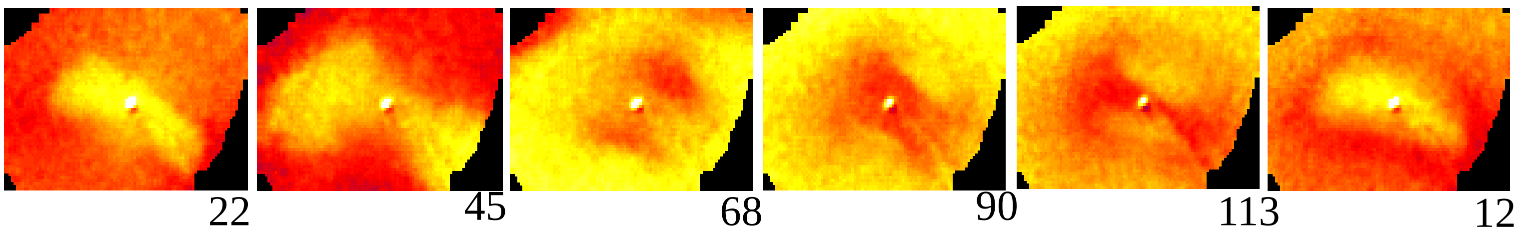
Spiral Wave, Figure-of-Eight, and Quatrefoil Reentry

- Spiral Wave (A)
  - S1 vert line
  - S2 horiz line
  - One singularity
- Figure-of-Eight (B)
  - S1 vert line
  - S2 point
  - Two singularities
- Quatrefoil (C & D)
  - Anisotropic cable
  - S1 Point
  - S2 Point
  - Cathodal (C) or anodal (D) have opposite rotations
  - Four singularities

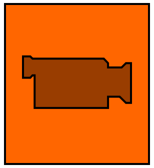


Optical Imaging of Quatrefoil Reentry

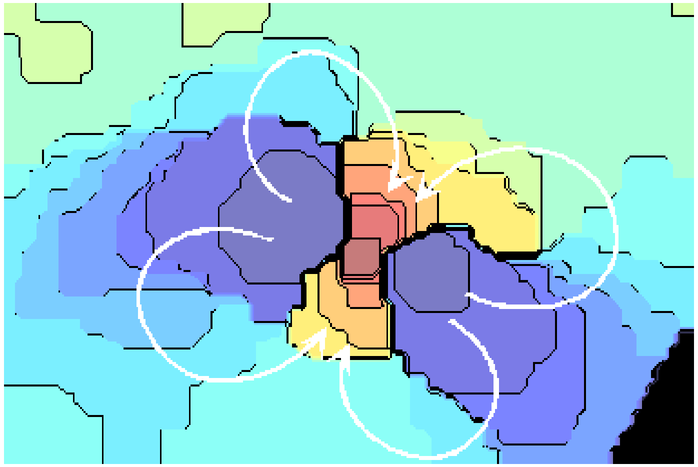
Transmembrane potential distributions from selected frames of a movie for cathodal-break stimulation in the isolated rabbit heart



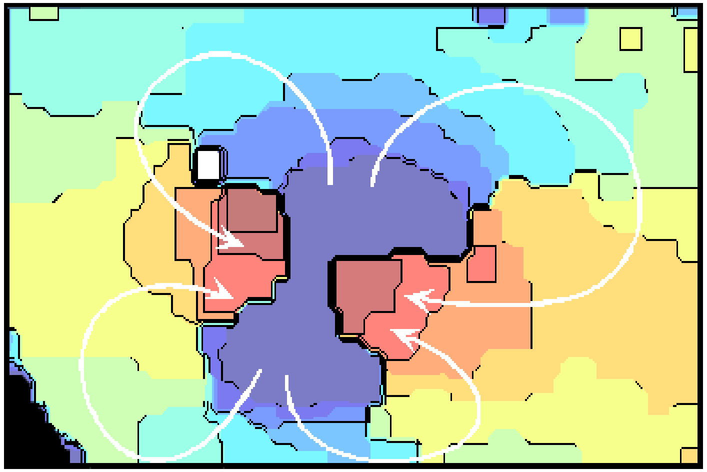
Vm_expt.mpg

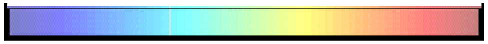


Cathodal-Break Isochrones



Anodal-Break Isochrones





Time →

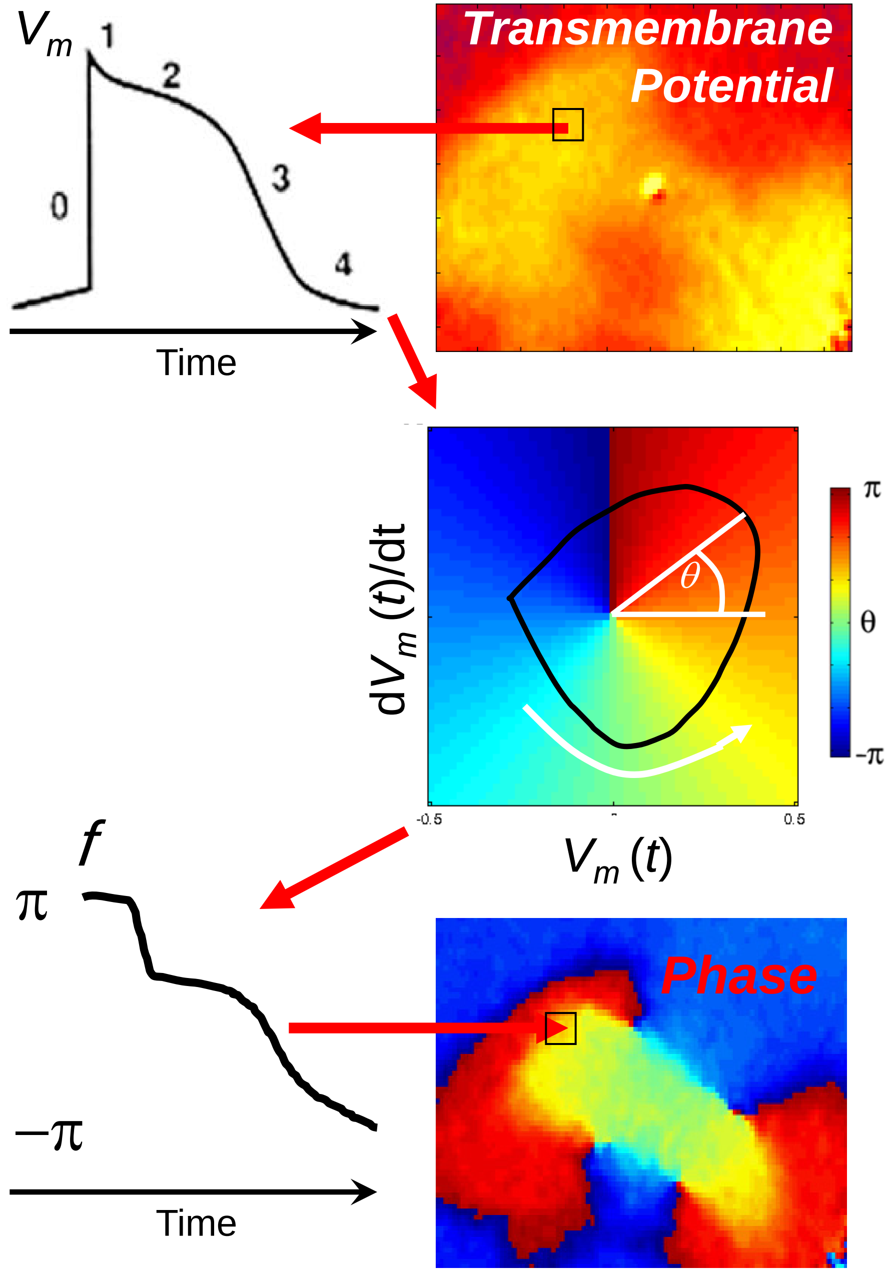
Transform into Phase Space

<u>The problem</u>: a given voltage can either be rising or falling

<u>The solution</u>: represent the cardiac action potential in terms of "phase" in the cardiac cycle

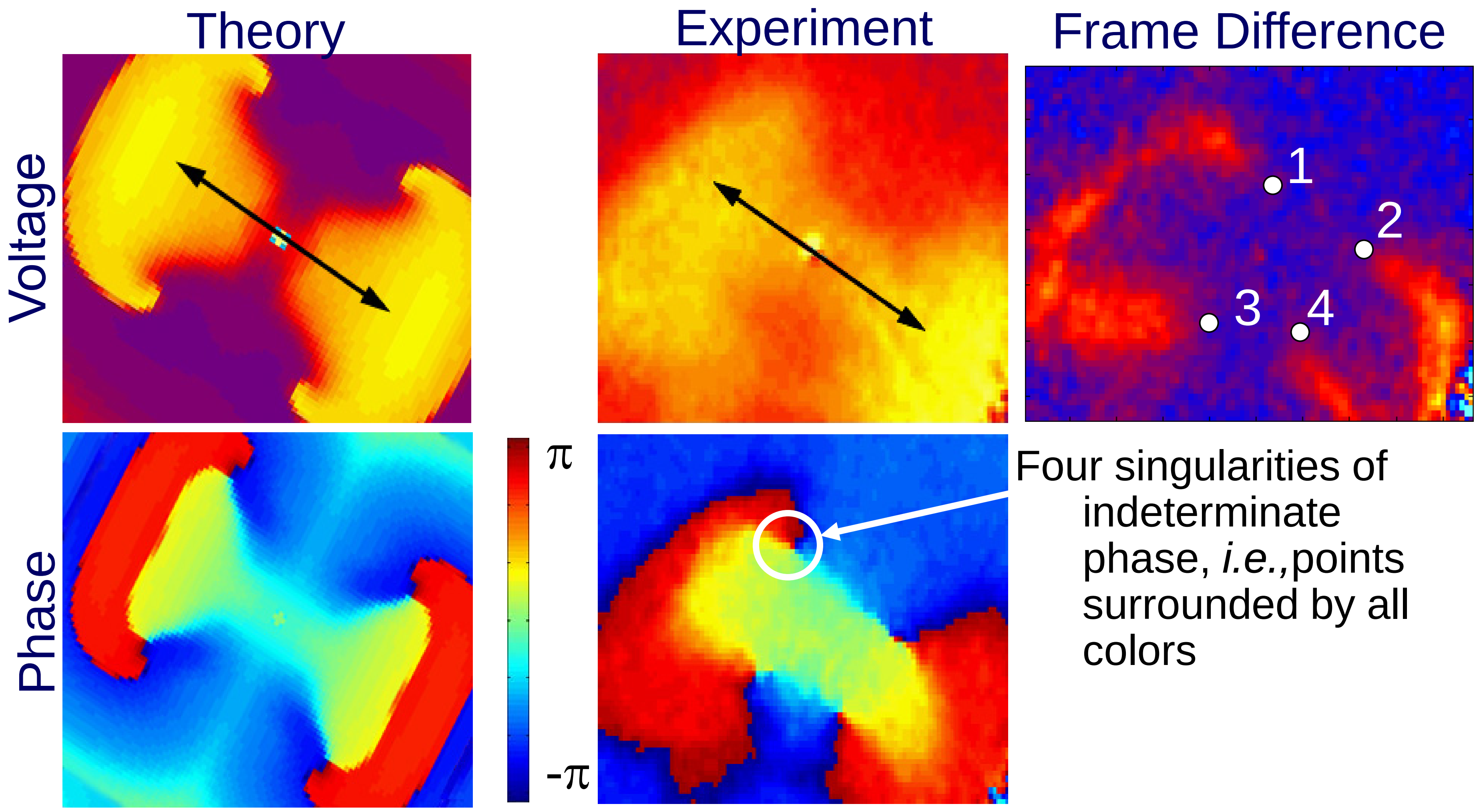
A standard definition of phase is

$$\phi(x,y,t) = \tan^{-1}\left[\frac{V_m(x,y,t)}{dV_m(x,y,t)/dt}\right]$$



Method by R.A. Gray, A.M. Pertsov, and J. Jalife, Nature **392:** 75 (1998).

From Voltage to Phase Space



Four singularities of indeterminate phase, *i.e.*,points surrounded by all colors

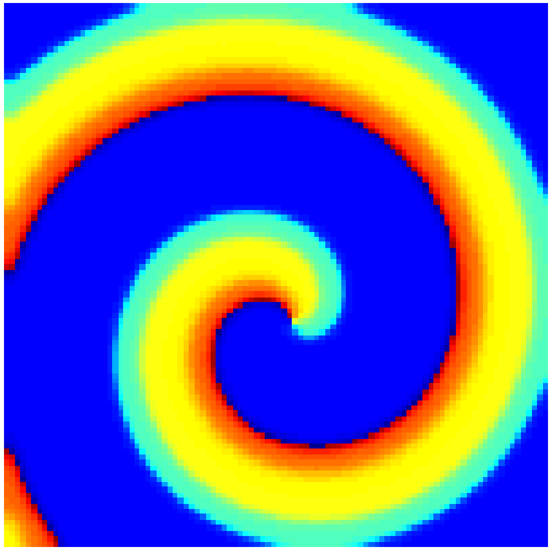
The Challenge

- Devise rapid image-processing algorithms to identify phase singularities in 1000 to 5000 frame-per-second movies of cardiac reentry
- Prove the relationship of the curl of the wave vector to topological charge
- Develop techniques to quantify the changes in shape of spiral waves due to parameter gradients and interactions with adjacent singularities

Topological Charge

$$n_t \equiv \frac{1}{2\pi} \oint_c \nabla\phi \cdot \vec{d\ell} \quad ,$$

$$n_t \equiv \frac{1}{2\pi} \oint_c \vec{k} \cdot \vec{d\ell} \quad ,$$



Phase(x,y)

- Topological charge n_t is zero about any closed path that does not encircle a phase singularity
- n_t is +1 or -1 for a path that encircles a singularity with a single arm
- Topological charge is conserved, *i.e.*, singularities are created and destroyed in pairs.

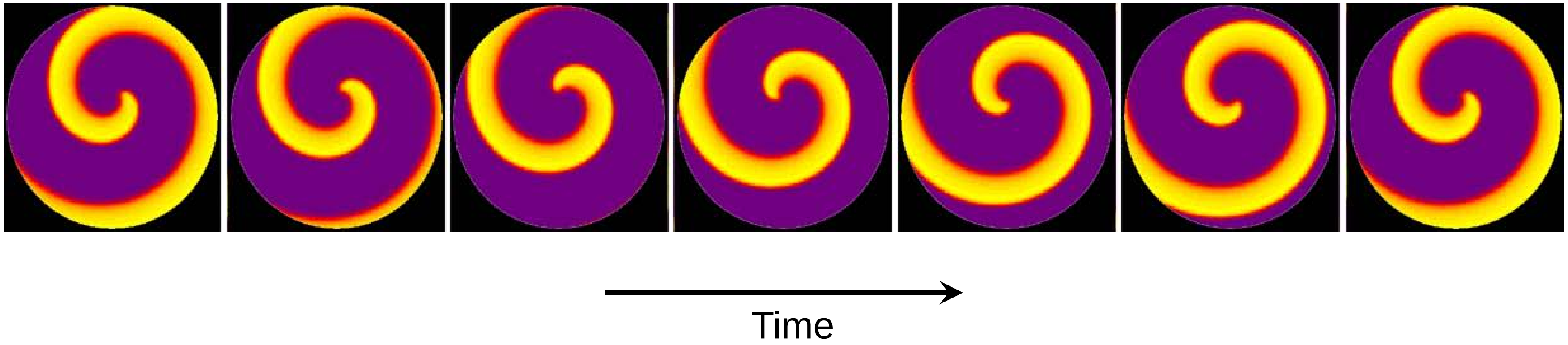
Simulated Wave Reentry

FitzHugh-Nagumo reaction-diffusion equations

$$\dot{v} = D_v \nabla^2 v + (\alpha - v)(v - 1)v - w,$$

$$\dot{w} = \varepsilon(\beta v - w),$$

and solve for $v(x,y,t)$



The Spiral Wave as a Wave Equation

- Consider a 2-D stationary sinusoidal wave with *q* periods circulating around the origin

$$\sin[\omega t + q \tan^{-1}(y/x)]$$

- We can express a spiral wave of arbitrary shape as a circular Fourier series,

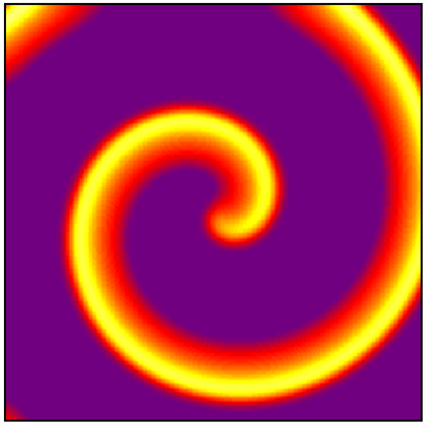
$$v(\vec{r},t) = \frac{A_0(r)}{2} + \sum_{\ell=1}^{\infty} A_\ell(r)\sin[\ell(\omega t + q\tan^{-1}(y/x))] + B_\ell(r)\cos[\ell(\omega t + q\tan^{-1}(y/x))]$$

$$A_\ell(r) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \alpha_k^s \sin k\omega_r r + \alpha_k^c \cos k\omega_r r,$$

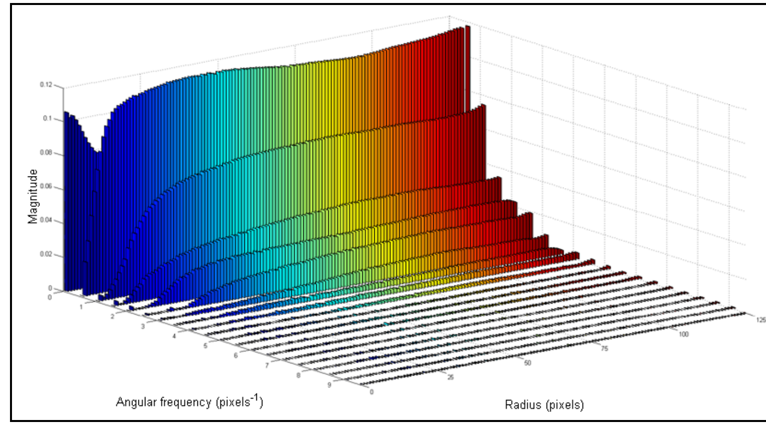
$$B_\ell(r) = \frac{\beta_0}{2} + \sum_{k=1}^{\infty} \beta_k^s \sin k\omega_r r + \beta_k^c \cos k\omega_r r,$$

Fourier Decomposition

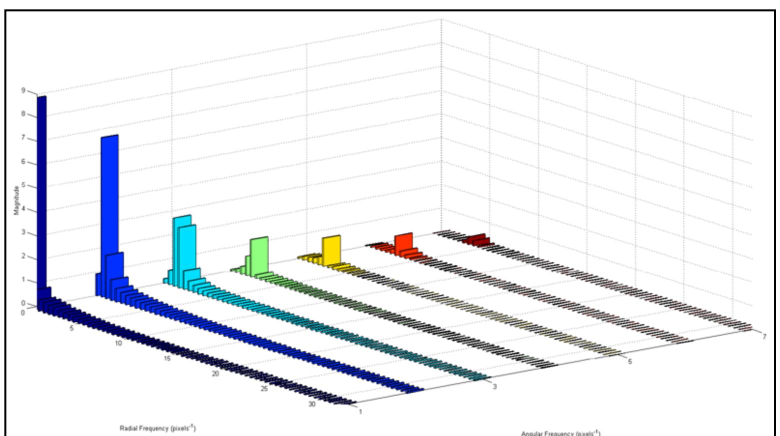
v



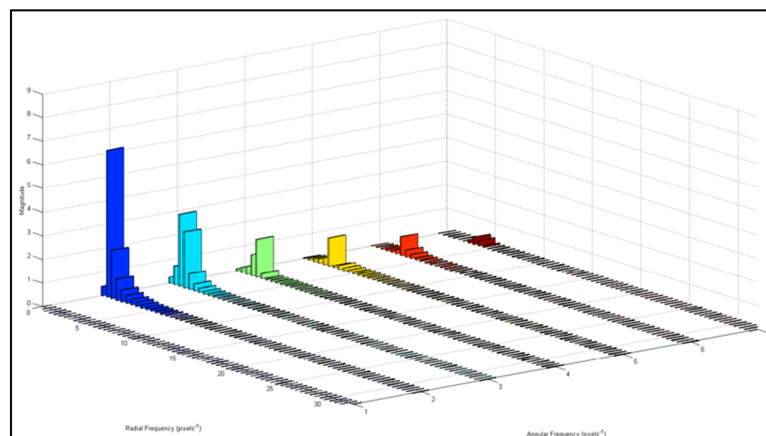
FFT →



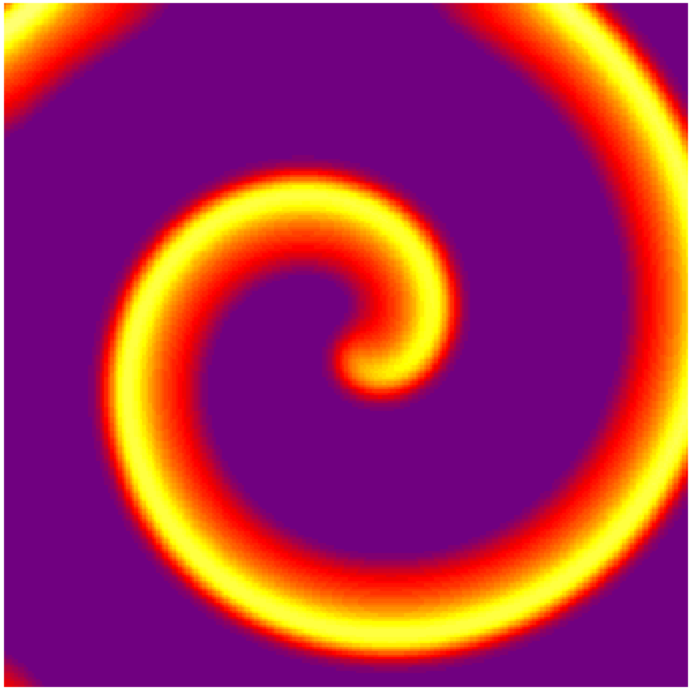
FFT_{real}

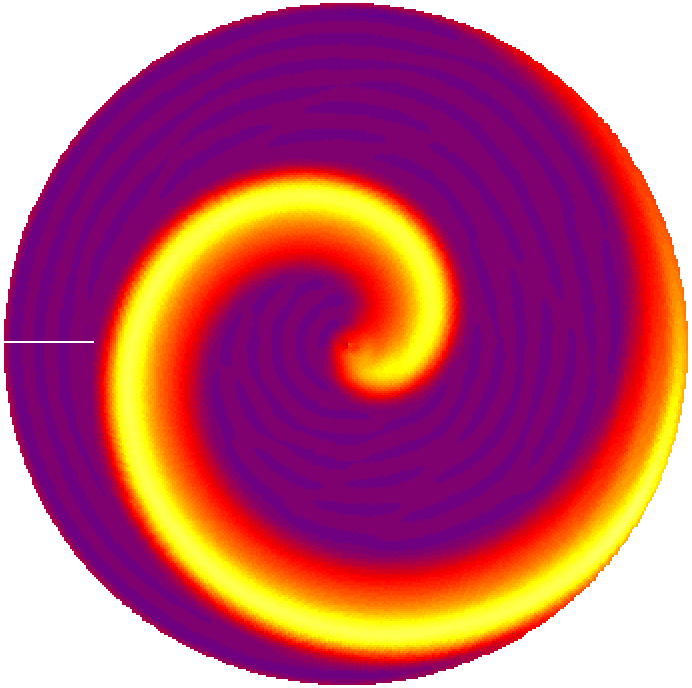


FFT_{imag}



Fourier Reconstruction





- Reconstruction to 10% error: 10 harmonics in angular direction, 12 in radial direction

Local Phase and the Wave Vector $\vec{k}$

- The spatial gradient of the phase ϕ is the wave vector $\vec{k}$.

$$\vec{k} = -\nabla\phi(x, y)$$

Topological Charge $\vec{k}$

$$n_t \equiv \frac{1}{2\pi} \oint_C \nabla\phi \cdot \vec{d\ell} \quad ,$$

$$n_t \equiv \frac{-1}{2\pi} \oint_C \vec{k} \cdot \vec{d\ell} \quad ,$$

Phase and Topological Charge

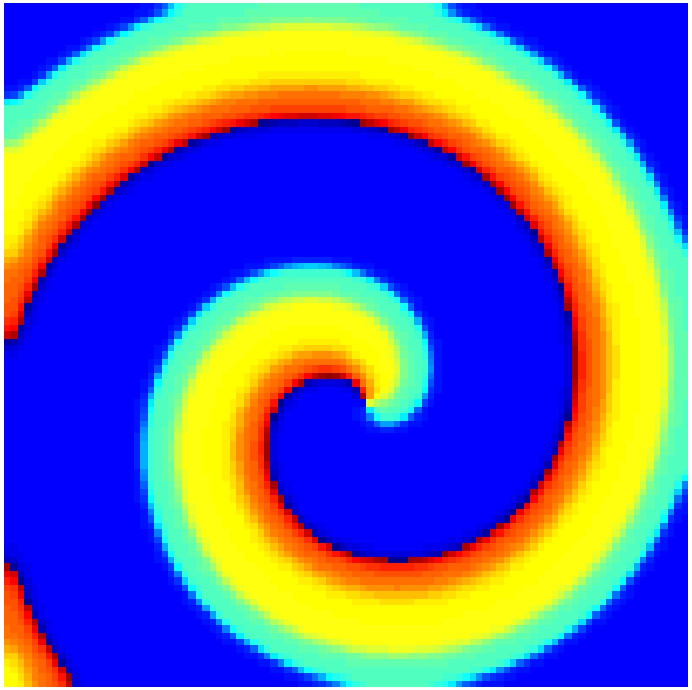
- Our definition of phase becomes the arctangent of the quotient of two sine/cosine series
- *Curl k* is proportional to the topological charge!

$$\hat{z}\cdot[\nabla\times\vec{k}(\vec{x})]=\frac{\partial k_y}{\partial x}-\frac{\partial k_x}{\partial y}=\lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint_c \vec{k}(\vec{r})\cdot d\vec{\ell}$$

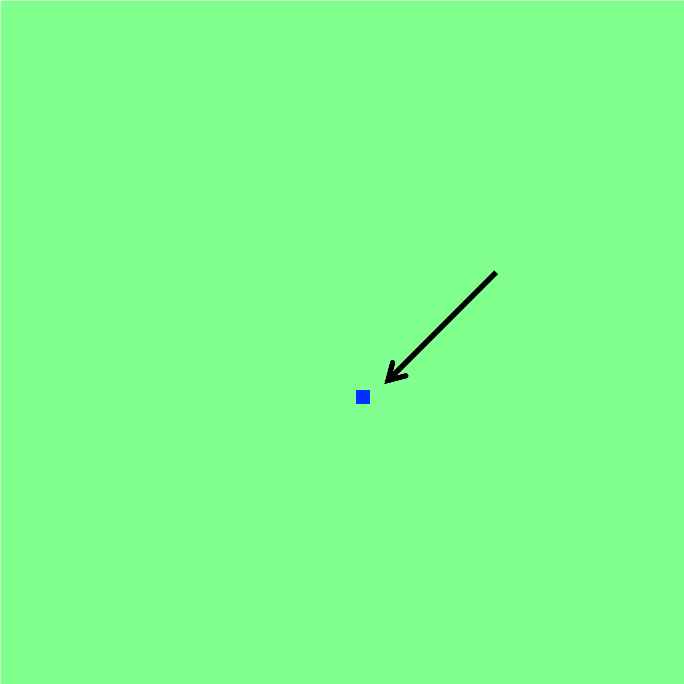
- It can be shown that the differential curl evaluates as exactly zero, except at the singularity where it is undefined
- At the singularity, the line integral around the singularity must be used directly to find the topological charge

Phase Singularities in Cardiac Reentry

Phase (ϕ) plot



$Curl\ k = Curl\ (\nabla\phi)$



The phase singularities can be identified by computing the curl of the gradient of the phase distribution

Topological charge

- *Curl k* may be approximated by

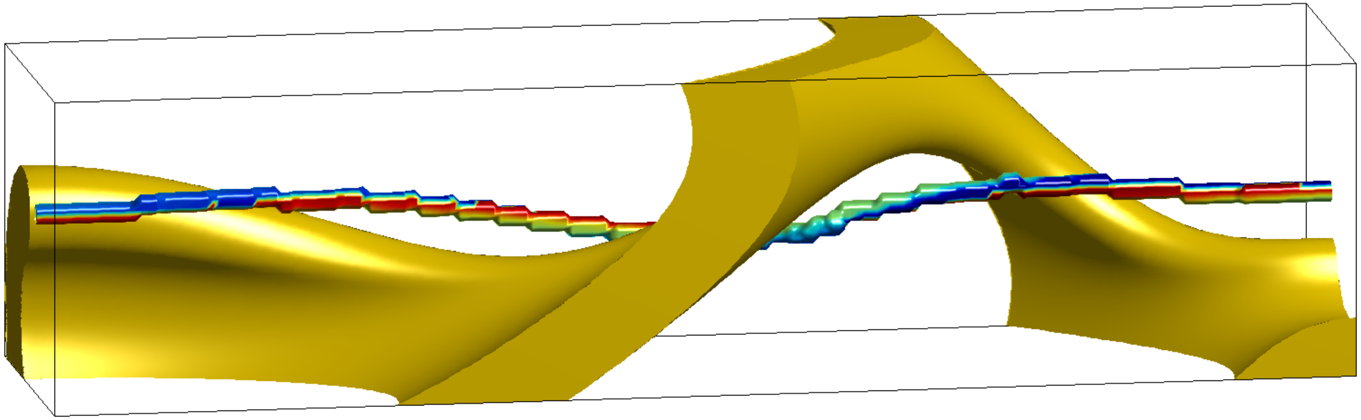
  1) a differential operator, or

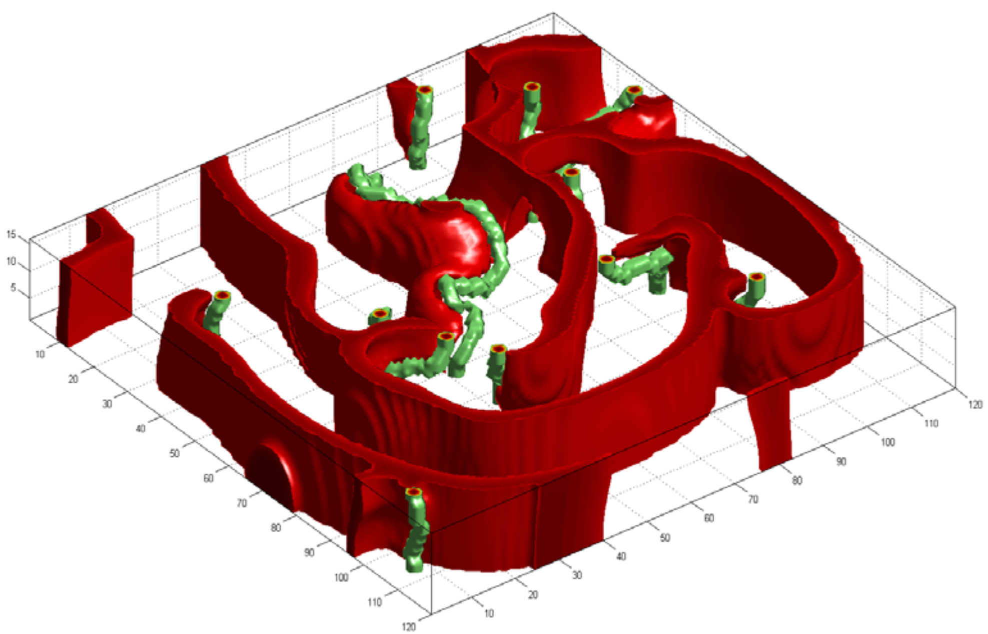
  2) as a discretized contour interval that is in fact a convolution operation of an image with two Nabla windows

$$\left(\nabla \times \vec{k}\right) \cdot \hat{z} \propto \nabla_x \otimes k_y + \nabla_y \otimes k_x,$$

$$\nabla_x = \begin{bmatrix} +1 & +1 & +1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix} \quad \nabla_y = \begin{bmatrix} -1 & 0 & +1 \\ -1 & 0 & +1 \\ -1 & 0 & +1 \end{bmatrix}$$

3-D Filaments

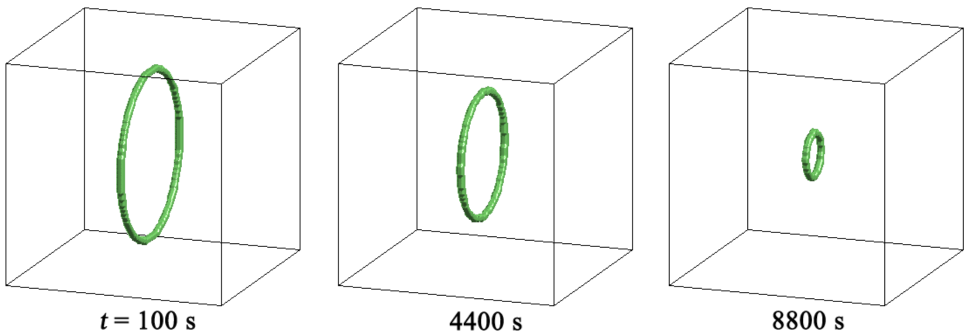




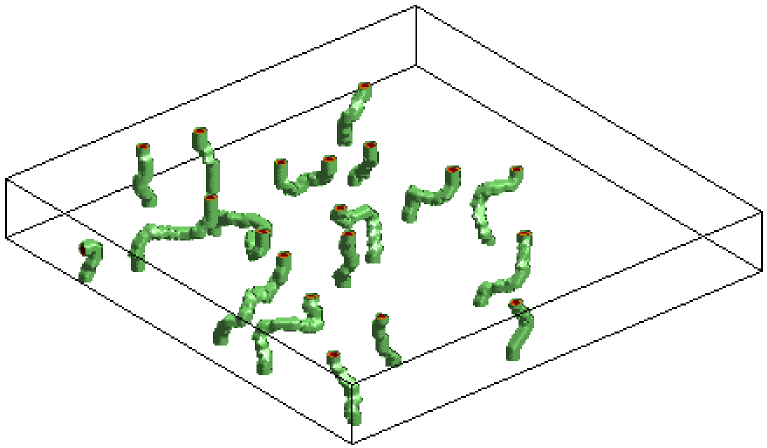
Because curl is a three-dimensional vector operator, this methodology may be extended readily to 3-D in order to visualize scroll wave filaments

Visualization of Filament Activity

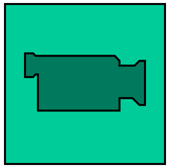
- The elegance of the approach is that $\nabla \times (\nabla f) = 0$ except at the singular filament!



Scroll ring shrinkage



Unstable scroll wave breakup



Scroll wave breakup

The Future

- In cardiac reentry, it is not uncommon to have many interacting spiral waves present; there is a need for a full mathematical analysis and description of the interaction between spiral waves
- The DC term in the Fourier decomposition is relatively constant at a distance from the phase singularity but exhibits an increase in the vicinity of the singularity
- Examination of this term for both the fast and slow variables may allow us to specify an *interaction potential* between two adjacent singularities, rather than having to integrate over the extent of both waves